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Numerical solution procedures for nonlinear elastic rods using the theory of a Cosserat point

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Abstract

The theory of a Cosserat point is developed as a continuum model, which is inherently nonlinear and is valid for arbitrary constitutive equations. Here, attention is confined to nonlinear elastic response, which is hyperelastic with a strain energy function, but large displacements, deformations and rotations are allowed. It is shown that the theory of a Cosserat point can be used to formulate a numerical solution procedure for the dynamic three-dimensional motion of nonlinear curved rods by modeling the rod as a set of N connected Cosserat points (like finite elements). Specifically, the Cosserat model allows for axial extension, tangential shear deformation, normal cross-sectional extension, normal cross-sectional shear deformation and rotary inertia. The Cosserat approach ensures that the global forms of the balances of linear and angular momentum are satisfied and the hyperelastic nature of the constitutive equations is preserved, since the response functions are determined by derivatives of a strain energy function. A number of static example problems have been considered, which examine the influence of shear deformation by comparing Cosserat solutions with nonlinear solutions of an elastica. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

A rod-like structure is a three-dimensional body that is essentially a space curve with some small cross-sectional area. The analysis of large deformations and large rotations of rods is of continued interest because such structures can be used to model flexible robotic arms, helicopter blades, DNA strands, polymer chains, etc. The early works of Kirchhoff and Euler, who analyzed large deformations of elastic rods in equilibrium, have been discussed by Love (1944). Within the context of these formulations, the reference curve of the rod is taken to be inextensible and the cross-sections are presumed to remain rigid and perpendicular to the tangent to this reference curve. Antman (1972,1974) has generalized this formulation of rod theory to include extension of the reference curve and tangential shear deformation, but the material fibers in the cross-section remain rigid.

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Using the theory of a Cosserat curve with two director vectors, Green et al. (1974a,b) have developed a more general rod theory that includes extension of the reference curve, tangential shear deformation, normal cross-sectional extension, and normal cross-sectional shear deformation (these deformations are described in detail later in this section). Based on this model, a hierarchy of constrained theories which eliminate one or more of these deformation modes has been discussed by Naghdi and Rubin (1984). Also, a simpler intrinsic formulation of a generalized Bernoulli–Euler type rod theory has recently been considered by Rubin (1997).

The equations of motion for rods are nonlinear partial differential equations, which are functions of one space variable and time. Consequently, for static problems, the equations become ordinary differential equations, which can be integrated using standard techniques like the shooting method to satisfy boundary conditions. In contrast, for dynamical problems, it is necessary to introduce some numerical procedure which discretizes the equations.

It is well known that the constitutive equations of a rod theory necessarily require coupling of the geometry of the rod-like structure and the material properties of the three-dimensional material from which the structure is made. For example, the typical coefficient E^*I^* of the bending moment in simple beam theory depends on the product of a material constant (Young's modulus of elasticity E^*) and a geometric constant (the second moment of area of the cross-section I^*). In this regard, from a constitutive point of view, the more general Cosserat theory with two deformable directors becomes simpler than theories in which the cross-section is rigid because it is possible to develop restrictions on the constitutive equations for nonlinear elastic rods that use the three-dimensional strain energy function and ensure consistency between the solutions of the rod theory and those of the three-dimensional theory for all homogeneous deformations (Rubin, 1996). Simo et al. (1990) also noted that the three-dimensional constitutive equation can be used in their numerical formulation of Cosserat-type shell theory when the director is deformable.

Simo (1985) has discussed a convenient parameterization of the rod model developed by Antman (1972), and Simo and Vu-Quoc (1986) have considered the associated finite element formulation. In this theory, the basic kinematic quantities are the position of a point on the reference curve and an orthogonal transformation that defines the rotation of an orthonormal triad that is attached to the cross-section at each point on the rod. The computational procedure uses a variational formulation of the equations of motion and an expansion of the kinematic quantities in terms of shape functions and nodal values. In particular, the constitutive equations for the rod theory are assumed to hold pointwise and the constitutive response of an individual element is obtained in the usual manner by integration.

The theory of a Cosserat point (Rubin, 1985a) is a special continuum theory that models deformation of a small structure that is essentially a point surrounded by some small but finite region. This theory has been used to formulate the numerical solution of problems in continuum mechanics (Rubin, 1985b,1986, 1987,1995) and it has been used by Green and Naghdi (1991) to model composite materials. Also, a unified treatment of constraints in the theory of a Cosserat point has been considered by O'Reilly and Vardi (1998). This work generalizes the notion of a Cosserat point to a collection of Cosserat points, which are connected by generalized constraints that can be explicit functions of time. Alternative theories for analyzing homogeneous deformations of zero-dimensional bodies have been developed by Slawianowski (1975,1982) and for pseudo-rigid bodies by Cohen (1981), Muncaster (1984) and Cohen and Muncaster (1984a,b).

Previous use of the theory of a Cosserat point for numerical solution procedures has been restricted to modeling elements that experience essentially homogeneous deformations. In contrast, here, the theory of a Cosserat point is used to model deformations of a rod element that can experience both general homogeneous deformation and inhomogeneous deformation associated with bending and torsion. Also, the initial shape of the rod can be curved.

Standard numerical methods for the equations of nonlinear hyperelasticity expand the position vector in terms of shape functions, which depend on the spatial variables only, and vector coefficients, which depend on time only (Finlayson and Scriven, 1966). For the method of weighted residuals (Petrov–Galerkin method), the local form of the balance of linear momentum is multiplied by weighting functions and then is integrated over the spatial region occupied by the structure to obtain a set of ordinary differential equations for the vector coefficients. If the weighting functions are the same as the shape functions, then this procedure is called the Bubnov–Galerkin method.

The degenerated shell approach developed by Ahmad et al. (1970) uses shape functions which are linear in the thickness coordinate. Within this context, it is possible to interpret the developments from the three-dimensional theory of the field equations: for Cosserat surfaces (Naghdi, 1972, Section 11) as a degenerated shell approach; for Cosserat rods (Green, et al., 1974a) as a degenerated rod approach; and for Cosserat points (Rubin, 1985a) as a degenerated point approach. If each element only experiences homogeneous deformation, then the Galerkin and the Cosserat approaches can be shown to be identical (except for possible differences in the director inertia coefficients; e.g. Rubin, 1985b; Rubin, 1995; Solberg and Papadopoulos, 1999). However, if the deformation in each element is allowed to be inhomogeneous, then the Galerkin and the Cosserat approaches can be different. This is because, within the context of the direct approach of the Cosserat theory, the constitutive equations for inhomogeneous deformations (like bending and torsion) are developed by comparison with known exact solutions or with appropriate experiments.

Specifically, for the numerical procedure developed here, the rod is divided into N elements that are connected through kinematic and kinetic conditions at their common ends. Each rod element is modeled as a Cosserat point. Fig. 1 shows the Ith element, which in its present configuration occupies the region of space ${}_IP$ that is bounded by the lateral surface ${}_I\partial P_{\rm L}$ and the two ends ${}_I\partial P_1$ and ${}_I\partial P_2$. Within the context of the theory, these ends are assumed to remain planar surfaces. The plane ${}_I\partial P_1$ is characterized by the two vectors ${}_I\mathbf{d}^*_\alpha(t)(\alpha = 1,2)$ and its centroid is located by the position vector ${}_I\mathbf{d}^*_0(t)$. Similarly, the plane ${}_I\partial P_2$ is characterized by the two vectors ${}_{I+1}\mathbf{d}^*_\alpha(t)$ and its centroid is located by the position vector ${}_I\mathbf{d}^*_0(t)$. It will be shown that ${}_I\mathbf{d}^*_i$ $(i = 0,1,2)$ represent nodal quantities in the proposed numerical procedure. Moreover, since ${}_{I+1}\mathbf{d}^*_\alpha(t)$ and ${}_{I+1}\mathbf{d}^*_\alpha$ are general vectors, the theory allows each of these cross-sections to experience tangential shear deformation (the bisector of the two vectors connecting the centroids of the elements on either sides of the node does not lie in the cross-sectional plane at that node), normal cross-sectional extension (the magnitudes of ${}_{I+1}\mathbf{d}^*_\alpha$ and ${}_{I+1}\mathbf{d}_\alpha*$ are not constant), and normal cross-sectional shear deformation (the angle between ${}_I\mathbf{d}^*_1$ and ${}_I\mathbf{d}^*_2$ and the angle between ${}_{I+1}\mathbf{d}^*_1$ and ${}_{I+1}\mathbf{d}^*_2$ are not constant). In particular, the physical importance of modeling normal cross-sectional extension in contact problems has been previously demonstrated (Naghdi and Rubin, 1989).

An outline of the main contents of this paper is as follows. Section 2 describes the balance laws of a single Cosserat point, Section 3 discusses superposed rigid body motions (SRBM), Section 4 presents

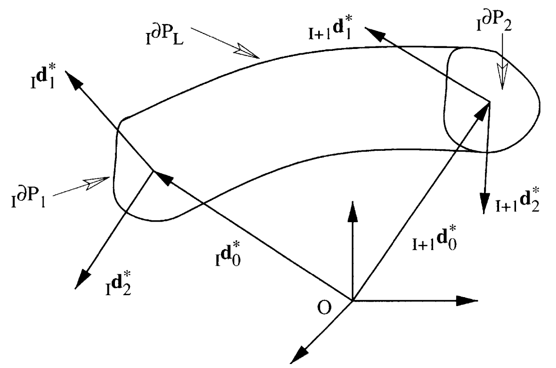

Fig. 1. Sketch of the Ith element that models a section of the rod.

constitutive equations for nonlinear elastic Cosserat points and Section 5 describes restrictions on the constitutive equations for homogeneous deformations. Section 6 describes an alternative reformulation of the balance laws, Section 7 discusses boundary conditions and Section 8 presents solutions to a set of problems that are used to determine the constitutive coefficients. Section 9 describes the numerical solution procedure for rod problems where the rod is modeled as a set of N connected Cosserat points. Then, Section 10 presents a number of examples which compare the Cosserat solution with that of the elastica, and which show the significant influence of shear deformation in some cases. Finally, Section 11 presents a brief summary of the work and Appendices A through E present relevant details of the developments.

Throughout the text, bold faced symbols are used to denote vector and tensor quantities. Also, $\mathbf{I}$ denotes the unity tensor; $\mathrm{tr}(\mathbf{A})$ denotes the trace of the second-order tensor $\mathbf{A}$; $\mathbf{A}^{\mathrm{T}}$ denotes the transpose of $\mathbf{A}$; $\mathbf{A}^{-1}$ denotes the inverse of $\mathbf{A}$; $\mathbf{A}^{-\mathrm{T}}$ denotes the inverse of the transpose of $\mathbf{A}$; and $\det(\mathbf{A})$ denotes the determinant of $\mathbf{A}$. The scalar $\mathbf{a} \cdot \mathbf{b}$ denotes the dot product between two vectors $\mathbf{a}$,$\mathbf{b}$; the scalar $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}\mathbf{B}^{\mathrm{T}})$ denotes the dot product between two second-order tensors $\mathbf{A}$,$\mathbf{B}$; the vector $\mathbf{a} \times \mathbf{b}$ denotes the cross product between $\mathbf{a}$ and $\mathbf{b}$; and the second order tensor $\mathbf{a} \otimes \mathbf{b}$ denotes the tensor product between $\mathbf{a}$ and $\mathbf{b}$. Moreover, the usual summation convention over repeated lower cased indices is implied with the range of Greek indices always being (1,2). The range of Latin indices will usually be (1,2,3) but sometimes it will be (0,1,...,5). Consequently, the range will be explicitly stated whenever it is not clear from the context. Moreover, there is no sum implied when the indices are upper cased letters.

2. Balance laws of the Cosserat point by the direct approach

In this section, the balance laws of the theory of a Cosserat point for a rod-like element are developed by a direct approach. However, the same forms of these balance laws can be developed by using a kinematic assumption and integrating the full three-dimensional theory as is shown in Appendix A. Also, for simplicity of notation, the kinematic and kinetic quantities associated with the Ith Cosserat point that models the Ith section of the rod will be written without a subscript I. However, later the subscript I will be used to distinguish between neighboring Cosserat points that are being connected when the numerical solution of rod problems is formulated.

In its present configuration at time t, the Cosserat point occupies a three-dimensional region of space P that is bounded by the lateral surface ∂P_{L} and the two ends ∂P_1 and ∂P_2, which are assumed to be planes (Fig. 2). A motion of the Cosserat point is characterized by six vector functions of time only

$$\mathbf{d}_i = \mathbf{d}_i(t) \quad \text{for } i = 0, 1, \ldots, 5. \tag{1}$$

The vector $\mathbf{d}_0$ locates the position of the Cosserat point relative to a fixed origin, the three director vectors $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ are assumed to be linearly independent

$$d^{1/2} = \mathbf{d}_1 \times \mathbf{d}_2 \cdot \mathbf{d}_3 > 0, \tag{2}$$

and they represent homogeneous deformations of the Cosserat point. The two director vectors $(\mathbf{d}_4, \mathbf{d}_5)$ characterize the potentially nonuniform shape of the Cosserat point and they are used to model inhomogeneous deformations. Also, the velocities $\mathbf{w}_i$ are defined by

$$\mathbf{w}_i = \dot{\mathbf{d}}_i \quad \text{for } i = 0, 1, \ldots, 5, \tag{3}$$

where a superposed dot denotes time differentiation. Moreover, for the numerical procedure discussed in this paper, it is necessary to relate the vectors $\mathbf{d}_i$ to the vectors associated with the ends of the rod section shown in Fig. 1, but this will be addressed later in the text.

The Cosserat point is endowed with mass m, and constant director inertias y^{ij}, which satisfy the equations

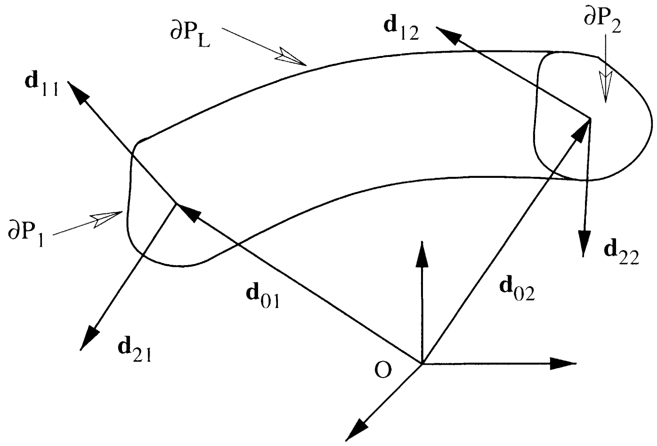

Fig. 2. Sketch of the vectors that define the Cosserat point in its present configuration.

$$y^{00} = 1, \qquad y^{ij} = y^{ji}, \qquad \dot{y}^{ij} = 0 \quad \text{for } i, j = 0, 1, \ldots, 5. \tag{4}$$

Using these quantities, the balance laws for the Cosserat point can be stated in the forms

$$\dot{m} = 0 \quad \text{or} \quad m = \rho d^{1/2} = \rho_0 D^{1/2},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{j=0}^{5} m y^{ij} \mathbf{w}_j\right] = m\mathbf{b}^i - \mathbf{t}^i \quad \text{for } i = 0, 1, \ldots, 5 \tag{5a, b}$$

with

$$\mathbf{t}^0 = 0, \qquad m\mathbf{b}^i = m\mathbf{B}^i + \mathbf{m}_1^i + \mathbf{m}_2^i \quad \text{for } i = 0, 1, \ldots, 5. \tag{6a, b}$$

In these equations, ρ is the mass density per unit present volume, ρ_0 is the mass density per unit reference volume, $D^{1/2}$ is the reference value of $d^{1/2}$, $\mathbf{B}^i$ represent the external forces and couples per unit mass applied to the Cosserat point by the body force and surface tractions on the lateral surface ∂P_{L}, $\mathbf{m}_\alpha^i$ represent the director couples applied to the ends ∂P_α, and $\mathbf{t}^i$ are the intrinsic director couples, which require constitutive equations. Eq. (5a) represent the conservation of mass, Eq. (5b) with $i = 0$ represents the balance of linear momentum, the remainder of Eqs. (5a,b) represent the balances of director momentum, and the balance of angular momentum can be stated in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=0}^{5}\sum_{j=0}^{5}[\mathbf{d}_i \times m y^{ij}\mathbf{w}_j] = \sum_{i=0}^{5}\mathbf{d}_i \times m\mathbf{b}^i, \tag{7}$$

where it is noted that the intrinsic director couples $\mathbf{t}^i$ do not contribute to the supply of angular momentum.

The Cosserat theory parallels the development of the three-dimensional theory in that the balance laws (5) are valid for all materials. Moreover, the constitutive equations are restricted by requiring the balance of angular momentum (7) to be satisfied for all materials and all deformations. In particular, it is noted that with the help of the balance laws (5), the reduced form of the balance of angular momentum (7) can be written as

$$\sum_{i=1}^{5}\mathbf{d}_i \times \mathbf{t}^i = 0. \tag{8}$$

Next, by introducing the second order tensor $\mathbf{T}$

$$\mathbf{T} = d^{-1/2}\sum_{i=1}^{5}\mathbf{t}^i \otimes \mathbf{d}_i, \tag{9}$$

it can be shown that Eq. (8) requires $\mathbf{T}$ to be a symmetric tensor

$$\mathbf{T}^{\mathrm{T}} = \mathbf{T}, \tag{10}$$

which is a similar result to that of the three-dimensional theory that requires the Cauchy stress tensor to be symmetric (A.18).

Furthermore, it is convenient to define the kinetic energy, $\mathscr{K}$ and the rate of work, $\mathscr{W}$ done by external forces and moments acting on the Cosserat point by the expressions

$$\mathscr{K} = \sum_{i=0}^{5}\sum_{j=0}^{5}\left[\frac{1}{2}my^{ij}\mathbf{w}_i \cdot \mathbf{w}_j\right], \qquad \mathscr{W} = \sum_{i=0}^{5}m\mathbf{b}^i \cdot \mathbf{w}_i. \tag{11a, b}$$

Then, the mechanical power, $\mathscr{P}$ can be defined by

$$d^{1/2}\mathscr{P} = \mathscr{W} - \dot{\mathscr{K}}, \tag{12}$$

and the balance laws (5) can be used to rewrite this expression in the alternative form

$$d^{1/2}\mathscr{P} = \sum_{i=1}^{5}\mathbf{t}^i \cdot \mathbf{w}_i. \tag{13}$$

In the stress-free reference configuration, the values of $\mathbf{d}_i$ are denoted by the constant vectors $\mathbf{D}_i$ and the three vectors $(\mathbf{D}_1, \mathbf{D}_2, \mathbf{D}_3)$ are assumed to be linearly independent

$$D^{1/2} = \mathbf{D}_1 \times \mathbf{D}_2 \cdot \mathbf{D}_3 > 0. \tag{14}$$

Moreover, in the following, it is convenient to define the reciprocal vectors $\mathbf{D}^i$ and $\mathbf{d}^i$, such that

$$\mathbf{D}_i \cdot \mathbf{D}^j = \delta_1^j, \qquad \mathbf{d}_i \cdot \mathbf{d}^j = \delta_1^j \quad \text{for } i,j = 1,2,3, \tag{15}$$

where δ_1^j is the Kronecker delta symbol. Then, the nonsingular deformation tensor $\mathbf{F}$ and its determinant J can be defined by the formulas

$$\mathbf{F} = \sum_{i=1}^{3}\mathbf{d}_i \otimes \mathbf{D}^i, \qquad J = \det(\mathbf{F}) = \frac{d^{1/2}}{D^{1/2}} > 0, \tag{16}$$

and the rate of deformation tensor $\mathbf{L}$ can be defined such that

$$\begin{gathered}\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}, \qquad \mathbf{L} = \sum_{i=1}^{3}\mathbf{w}_i \otimes \mathbf{d}^i = \mathbf{D} + \mathbf{W}, \\ \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\mathrm{T}}) = \mathbf{D}^{\mathrm{T}}, \qquad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^{\mathrm{T}}) = -\mathbf{W}^{\mathrm{T}},\end{gathered} \tag{17}$$

where $\mathbf{D}$ and $\mathbf{W}$ are the symmetric and skew-symmetric parts of $\mathbf{L}$, respectively. Also, it will be shown later that the quantities $\boldsymbol{\beta}_\alpha$ defined by

$$\boldsymbol{\beta}_1 = \mathbf{F}^{-1}\mathbf{d}_4 - \mathbf{D}_4, \qquad \boldsymbol{\beta}_2 = \mathbf{F}^{-1}\mathbf{d}_5 - \mathbf{D}_5, \tag{18}$$

are strain measures that characterize inhomogeneous deformations of the Cosserat point.

Next, it is desirable to derive an alternative expression for the mechanical power in terms of the variables $\mathbf{L}$ and $\dot{\boldsymbol{\beta}}_\alpha$. To this end, it is first recalled that

$$\dot{\overline{\mathbf{F}^{-1}}} = -\mathbf{F}^{-1}\mathbf{L}, \qquad \mathbf{w}_i = \mathbf{L}\mathbf{d}_i \quad \text{for } i = 1,2,3. \tag{19}$$

Thus, the material derivatives of $\boldsymbol{\beta}_\alpha$ become

$$\dot{\boldsymbol{\beta}}_1 = \mathbf{F}^{-1}(\mathbf{w}_4 - \mathbf{L}\mathbf{d}_4), \qquad \dot{\boldsymbol{\beta}}_2 = \mathbf{F}^{-1}(\mathbf{w}_5 - \mathbf{L}\mathbf{d}_4), \tag{20}$$

which allow the mechanical power to be expressed in the form

$$d^{1/2}\mathscr{P} = d^{1/2}\mathbf{T}\cdot\mathbf{L} + (\mathbf{F}^{\mathrm{T}}\mathbf{t}^4)\cdot\dot{\boldsymbol{\beta}}_1 + (\mathbf{F}^{\mathrm{T}}\mathbf{t}^5)\cdot\dot{\boldsymbol{\beta}}_2. \tag{21}$$

Furthermore, with the help of the reduced form of the balance of angular momentum (10), this expression can be rewritten as

$$d^{1/2}\mathscr{P} = d^{1/2}\mathbf{T}\cdot\mathbf{D} + (\mathbf{F}^{\mathrm{T}}\mathbf{t}^4)\cdot\dot{\boldsymbol{\beta}}_1 + (\mathbf{F}^{\mathrm{T}}\mathbf{t}^5)\cdot\dot{\boldsymbol{\beta}}_2. \tag{22}$$

Moreover, within the context of the purely mechanical theory, the rate of material dissipation $\mathscr{D}$ can be defined by the expression

$$d^{1/2}\mathscr{D} = d^{1/2}\mathscr{P} - m\dot{\Sigma} = \mathscr{W} - \dot{\mathscr{K}} - m\dot{\Sigma} \geqslant 0, \tag{23}$$

where Σ is the strain energy function per unit mass. Also, constitutive equations for $\mathbf{T}$ and $\mathbf{t}^i$ must be restricted so that the rate of material dissipation remains nonnegative.

It will be shown later that for three-dimensionally homogeneous deformations, the tensor $\mathbf{F}$ is equal to the three-dimensional deformation gradient tensor. This result suggests that it is convenient to introduce the tensors $\mathbf{C}$ and $\mathbf{B}$ by the expressions

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}, \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}, \tag{24}$$

so that $\mathbf{C}$ is similar to the right Cauchy–Green deformation tensor and $\mathbf{B}$ is similar to the left Cauchy–Green deformation tensor. Also, by differentiating these expressions with respect to time and using Eq. (17), it can be shown that

$$\dot{\mathbf{C}} = 2\mathbf{F}^{\mathrm{T}}\mathbf{D}\mathbf{F}, \qquad \dot{\mathbf{B}} = \mathbf{L}\mathbf{B} + \mathbf{B}\mathbf{L}^{\mathrm{T}}. \tag{25}$$

3. Superposed rigid body motions

For a complete theory, it is necessary to discuss how the kinematic and kinetic quantities transform under SRBM. To this end, it is noted that under SRBM, the position vector $\mathbf{d}_0$ of the Cosserat point in its present configuration at time t is transformed to the position vector $\mathbf{d}_0^+$ at time t^+ in the superposed configuration such that

$$\mathbf{d}_0^+ = \mathbf{c}(t) + \mathbf{Q}(t)\mathbf{d}_0, \qquad t^+ = t + a, \tag{26}$$

where $\mathbf{c}(t)$ is an arbitrary vector function of time only that characterizes the translation of the Cosserat point, $\mathbf{Q}(t)$ is a proper orthogonal tensor function of time only

$$\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}, \qquad \det\mathbf{Q} = +1 \tag{27a, b}$$

that characterizes rotation, and a is a constant time shift. Furthermore, by differentiating Eq. (27a) with respect to time, it can be shown that

$$\dot{\mathbf{Q}} = \boldsymbol{\Omega}\mathbf{Q}, \qquad \boldsymbol{\Omega}^{\mathrm{T}} = -\boldsymbol{\Omega}, \tag{28}$$

where $\boldsymbol{\Omega}(t)$ is a skew-symmetric tensor function of time. Also, the director vectors transform by

$$\mathbf{d}_i^+ = \mathbf{Q}(t)\mathbf{d}_i \quad \text{for } i = 1, 2, \ldots, 5. \tag{29}$$

In these equations and throughout the text, the value of a quantity in the superposed configuration is denoted by the same symbol as in the present configuration, but with an added superposed (+).

Now, with the help of these kinematical relations, it is possible to determine transformation relations for all other kinematical quantities. For example, it can be shown that

$$\begin{aligned} &\mathbf{d}^{i+} = \mathbf{Q}\mathbf{d}^i \quad \text{for } i = 1, 2, 3, \qquad \boldsymbol{\beta}_\alpha^+ = \boldsymbol{\beta}_\alpha, \\ &\mathbf{F}^+ = \mathbf{Q}\mathbf{F}, \qquad J^+ = \det(\mathbf{F}^+) = J, \\ &\mathbf{C}^+ = \mathbf{C}, \qquad \mathbf{B}^+ = \mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}}, \\ &\mathbf{L}^+ = \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}} + \boldsymbol{\Omega}, \qquad \mathbf{D}^+ = \mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathrm{T}}, \qquad \mathbf{W}^{\mathrm{T}} = \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}} + \boldsymbol{\Omega}. \end{aligned} \tag{30}$$

To determine how the kinetic quantities transform under SRBM, it is assumed that the balance laws remain unchanged in form. However, to determine how $\mathbf{B}^i$, $\mathbf{t}^i$, $\mathbf{m}_\alpha^i$ transform individually, it is necessary to make additional assumptions. In this regard, it is recalled that within the context of the three-dimensional theory, assumptions are made which lead to the fact that the traction vector acting on an arbitrary surface is merely rotated under SRBM. Thus, for the theory of a Cosserat point it is assumed that the kinetic quantities transform by

$$m^+ = m, \qquad \rho^+ = \rho_0^+, \qquad y^{ij+} = y^{ij}, \tag{31a, b, c}$$

$$\mathbf{B}^{i+} - \sum_{j=0}^{5} m^+ y^{ij+} \dot{\mathbf{w}}_j^+ = \mathbf{Q}\left[m\mathbf{B}^i - \sum_{j=0}^{5} m y^{ij} \dot{\mathbf{w}}_j, \right] \tag{31d}$$

$$\mathbf{t}^{i+} = \mathbf{Q}\mathbf{t}^i, \quad \mathbf{m}_\alpha^{i+} = \mathbf{Q}\mathbf{m}_\alpha^i \quad \text{for } i = 0, 1, 2, \ldots, 5, \tag{31e, f}$$

where Eqs. (31e,f) have been motivated by the three-dimensional condition on the traction vector. Also, the mechanical power $\mathscr{P}$ and the strain energy function Σ are assumed to remain unaltered by SRBM

$$\mathscr{P}^+ = \mathscr{P}, \qquad \Sigma^+ = \Sigma. \tag{32}$$

4. A nonlinear elastic Cosserat point

For a nonlinear elastic Cosserat point, the strain energy function Σ is assumed to depend on the variables $(\mathbf{F}, \boldsymbol{\beta}_\alpha)$ as well as on the reference vectors $\mathbf{D}_i$ and the reference geometry of the Cosserat point. However, since Σ must remain unaltered under SRBM, it can be shown that Σ must depend on $\mathbf{F}$ only through the deformation tensor $\mathbf{C}$. Consequently, for an elastic Cosserat point, the strain energy function takes the form

$$\Sigma = \hat{\Sigma}(\mathbf{C}, \boldsymbol{\beta}_\alpha). \tag{33}$$

Thus, with the help of Eqs. (5), (22) and (25), the rate of dissipation (23) can be expressed as

$$d^{1/2}\mathscr{D} = d^{1/2}\left[\mathbf{T} - 2\rho\mathbf{F}\frac{\partial\hat{\Sigma}}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}}\right] \cdot \mathbf{D} + \left[\mathbf{F}^{\mathrm{T}}\mathbf{t}^4 - m\frac{\partial\hat{\Sigma}}{\partial\boldsymbol{\beta}_1}\right] \cdot \dot{\boldsymbol{\beta}}_1 + \left[\mathbf{F}^{\mathrm{T}}\mathbf{t}^5 - m\frac{\partial\hat{\Sigma}}{\partial\boldsymbol{\beta}_2}\right] \cdot \dot{\boldsymbol{\beta}}_2. \tag{34}$$

Moreover, for purely elastic response, it is assumed that the kinetic quantities $(\mathbf{T}, \mathbf{t}^i)$ are given by functions $(\hat{\mathbf{T}}, \hat{\mathbf{t}}^i)$, which are independent of the rates $(\mathbf{D}, \dot{\boldsymbol{\beta}}_\alpha)$

$$\mathbf{T} = \hat{\mathbf{T}}, \qquad \mathbf{t}^i = \hat{\mathbf{t}}^i \quad \text{for } i = 1, 2, \ldots, 5, \tag{35}$$

and that the dissipation $\mathscr{D}$ vanishes for all deformations so that

$$\hat{\mathbf{T}} = 2\rho \mathbf{F} \frac{\partial \hat{\Sigma}}{\partial \mathbf{C}} \mathbf{F}^{\mathrm{T}}, \qquad \hat{\mathbf{t}}^4 = \mathbf{F}^{-\mathrm{T}} m \frac{\partial \hat{\Sigma}}{\partial \boldsymbol{\beta}_1}, \qquad \hat{\mathbf{t}}^5 = \mathbf{F}^{-\mathrm{T}} m \frac{\partial \hat{\Sigma}}{\partial \boldsymbol{\beta}_2}. \tag{36}$$

Also, using definition (9) and the reciprocal vectors $\mathbf{d}^i$, it follows that

$$\hat{\mathbf{t}}^i = [d^{1/2}\hat{\mathbf{T}} - \hat{\mathbf{t}}^4 \otimes \mathbf{d}_4 - \hat{\mathbf{t}}^5 \otimes \mathbf{d}_5] \cdot \mathbf{d}^i \quad \text{for } i = 1, 2, 3. \tag{37}$$

More general material response with dissipation can be modeled by assuming that $(\mathbf{T}, \mathbf{t}^i)$ separate additively into elastic parts $(\hat{\mathbf{T}}, \hat{\mathbf{t}}^i)$, which are given by Eq. (36), and dissipative parts $(\check{\mathbf{T}}, \check{\mathbf{t}}^i)$ such that

$$\mathbf{T} = \hat{\mathbf{T}} + \check{\mathbf{T}}, \qquad \mathbf{t}^i = \hat{\mathbf{t}}^i + \check{\mathbf{t}}^i. \tag{38}$$

Then, the dissipation (34) reduces to

$$\mathscr{D} = \check{\mathbf{T}} \cdot \mathbf{D} + d^{-1/2}(\mathbf{F}^{\mathrm{T}}\check{\mathbf{t}}^4) \cdot \dot{\boldsymbol{\beta}}_1 + d^{-1/2}(\mathbf{F}^{\mathrm{T}}\check{\mathbf{t}}^5) \cdot \dot{\boldsymbol{\beta}}_2 \geqslant 0, \tag{39}$$

which is required to be nonnegative. Furthermore, with the help of Eq. (9), it follows that

$$\check{\mathbf{t}}^i = [d^{1/2}\check{\mathbf{T}} - \check{\mathbf{t}}^4 \otimes \mathbf{d}_4 - \check{\mathbf{t}}^5 \otimes \mathbf{d}_5] \cdot \mathbf{d}^i \quad \text{for } i = 1, 2, 3. \tag{40}$$

As a special case, these dissipative parts can be expressed in the simple forms

$$\begin{gathered} \check{\mathbf{T}} = D^{1/2} V[\eta_1(\mathbf{D} \cdot \mathbf{I})\mathbf{I} + 2\eta_2 \mathbf{D}'], \quad \mathbf{D}' = \mathbf{D} - \tfrac{1}{3}(\mathbf{D} \cdot \mathbf{I})\mathbf{I}, \quad \mathbf{D}' \cdot \mathbf{I} = 0, \\ \check{\mathbf{t}}^4 = \eta_4 D^{1/2} V(\mathbf{D}^1 \cdot \mathbf{D}^1)\mathbf{F}^{-\mathrm{T}}\dot{\boldsymbol{\beta}}_1, \quad \check{\mathbf{t}}^5 = \eta_5 D^{1/2} V(\mathbf{D}^2 \cdot \mathbf{D}^2)\mathbf{F}^{-\mathrm{T}}\dot{\boldsymbol{\beta}}_2, \end{gathered} \tag{41}$$

where V is related to the volume of the element (see Eq. (C.9)), $\mathbf{D}'$ is a pure measure of distortional deformation rate and the four material constants $(\eta_1, \eta_2, \eta_4, \eta_5)$ must be nonnegative

$$\eta_1 \geqslant 0, \quad \eta_2 \geqslant 0, \quad \eta_4 \geqslant 0, \quad \eta_5 \geqslant 0, \tag{42}$$

in order for the dissipation to be nonnegative for all deformations.

5. Restrictions for homogeneous deformations

In general, the constitutive equations for $\mathbf{T}$ and $\mathbf{t}^i$ in the theory of a Cosserat point depend on both the material properties of the material that is used to construct the point-like structure and on the specific geometry of that structure. Even if the strain energy function Σ^* of the three-dimensional material, which is used to construct the rod section, is known, it is not trivial to determine the strain energy function Σ for the Cosserat point for general deformations. A similar problem occurs in the theories of shells and rods and a partial resolution of this problem has been developed by Naghdi and Rubin (1995) for shells and by Rubin (1996) for rods. Following that work, it is observed that if a rod-like structure is constructed from a nonlinear elastic material that is three-dimensionally uniform and homogeneous, then the constitutive equations for the Cosserat point theory can be suitably restricted so that predictions of the Cosserat theory will be consistent with exact solutions of the three-dimensional equations for all three-dimensionally homogeneous deformations. The objective of this section is to develop these restrictions. Also, in the remainder of this paper, it will be assumed that the material that is used to construct the Cosserat point is three-dimensionally homogeneous with constant mass density ρ_0^*.

To this end, it is first noted from Appendix C that the necessary and sufficient conditions for the deformation to be three-dimensionally homogeneous require the strains $\boldsymbol{\beta}_\alpha$ to vanish

$$\boldsymbol{\beta}_\alpha = 0. \tag{43}$$

This proves that the strains $\boldsymbol{\beta}_\alpha$ are natural measures of inhomogeneous deformation of the Cosserat point, as has already been stated. Under these conditions, the three-dimensional deformation gradient $\mathbf{F}^*$ becomes a function of time only that is equal to the tensor $\mathbf{F}$ defined in Eq. (16) (see Eq. (C.8))

$$\mathbf{F}^* = \mathbf{F}(t). \tag{44}$$

Moreover, for homogeneous deformations of a three-dimensionally homogeneous material, the mass m of the Cosserat point can be expressed in the form (C.9)

$$m = \rho_0^* V D^{1/2}, \tag{45}$$

where V is related to the volume of the Cosserat point in its reference configuration.

Next, by comparing the elastic constitutive equations (36) with the results (C.11) and using Eq. (5), it can be seen that the Cosserat point theory will be consistent with the three-dimensional theory for all homogeneous deformations if the strain energy function Σ satisfies the restrictions

$$\frac{\partial\hat{\Sigma}(\mathbf{C},\boldsymbol{\beta}_\alpha)}{\partial\mathbf{C}} = \frac{\partial\Sigma^*(\mathbf{C})}{\partial\mathbf{C}}, \qquad \frac{\partial\hat{\Sigma}(\mathbf{C},\boldsymbol{\beta}_\alpha)}{\partial\boldsymbol{\beta}_\alpha} = 2\mathbf{C}\frac{\partial\Sigma^*(\mathbf{C})}{\partial\mathbf{C}}\mathbf{V}^\alpha \quad \text{for } \boldsymbol{\beta}_\alpha = 0, \tag{46}$$

where $\mathbf{V}^\alpha$ are constant vectors defined by integrals (C.12).

It will presently be shown that these restrictions can be simplified by introducing the auxiliary deformation tensors $\overline{\mathbf{F}}$ and $\overline{\mathbf{C}}$ defined by

$$\begin{aligned} \overline{\mathbf{F}} &= \overline{\mathbf{F}}(\mathbf{C},\boldsymbol{\beta}_\alpha;\mathbf{V}^\alpha) = \mathbf{F}[\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{V}^\alpha], \\ \overline{\mathbf{C}} &= \overline{\mathbf{C}}(\mathbf{C},\boldsymbol{\beta}_\alpha;\mathbf{V}^\alpha) = \overline{\mathbf{F}}^{\mathrm{T}}\overline{\mathbf{F}} = [\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{V}^\alpha]^{\mathrm{T}}\mathbf{C}[\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{V}^\beta], \end{aligned} \tag{47}$$

which satisfy the conditions that for homogeneous deformations ($\boldsymbol{\beta}_\alpha = 0$)

$$\overline{\mathbf{F}}(\mathbf{C},0;\mathbf{V}^\alpha) = \mathbf{F}, \qquad \overline{\mathbf{C}}(\mathbf{C},0;\mathbf{V}^\alpha) = \mathbf{C}. \tag{48}$$

Now, provided that the inhomogeneous deformation is never great enough to cause $\overline{\mathbf{F}}$ to be singular

$$\det[\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{V}^\alpha] > 0, \tag{49}$$

it follows that a general form for the strain energy function Σ can be specified in terms of the three-dimensional strain energy function Σ^* by the expression

$$\Sigma = \Sigma^*(\overline{\mathbf{C}}) + \Psi(\mathbf{C},\boldsymbol{\beta}_\alpha), \tag{50}$$

where Ψ is an additive part of the strain energy due to inhomogeneous deformations that can depend on the reference geometry. Next, by taking the derivative of Σ, it can be shown that

$$\begin{aligned} \dot{\Sigma} &= \left[\{\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{V}^\alpha\}\frac{\partial\Sigma^*(\overline{\mathbf{C}})}{\partial\overline{\mathbf{C}}}\{\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{V}^\beta\}^{\mathrm{T}} + \frac{\partial\Psi}{\partial\mathbf{C}}\right]\cdot\dot{\mathbf{C}} + \left[2\mathbf{C}\{\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{V}^\beta\}\frac{\partial\Sigma^*(\overline{\mathbf{C}})}{\partial\overline{\mathbf{C}}}\mathbf{V}^\alpha + \frac{\partial\Psi}{\partial\boldsymbol{\beta}_\alpha}\right]\cdot\dot{\boldsymbol{\beta}}_\alpha, \\ \frac{\partial\Sigma}{\partial\mathbf{C}} &= \left[\{\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{V}^\alpha\}\frac{\partial\Sigma^*(\overline{\mathbf{C}})}{\partial\overline{\mathbf{C}}}\{\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{V}^\beta\}^{\mathrm{T}} + \frac{\partial\Psi}{\partial\mathbf{C}}\right], \quad \frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha} = \left[2\mathbf{C}\{\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{V}^\beta\}\frac{\partial\Sigma^*(\overline{\mathbf{C}})}{\partial\overline{\mathbf{C}}}\mathbf{V}^\alpha + \frac{\partial\Psi}{\partial\boldsymbol{\beta}_\alpha}\right]. \end{aligned} \tag{51}$$

Thus, the strain energy function (50) satisfies restrictions (46) provided that

$$\frac{\partial\Psi}{\partial\mathbf{C}} = 0, \qquad \frac{\partial\Psi}{\partial\boldsymbol{\beta}_\alpha} = 0 \qquad \text{for } \boldsymbol{\beta}_\alpha = 0. \tag{52}$$

Moreover, substitution of results (52) into Eq. (36) yields constitutive equations for the elastic parts $(\hat{\mathbf{T}},\hat{\mathbf{t}}^i)$ of the kinetic quantities $(\mathbf{T},\mathbf{t}^i)$ of the forms

$$d^{1/2}\hat{\mathbf{T}} = 2m\left[\overline{\mathbf{F}}\frac{\partial\Sigma^*(\overline{\mathbf{C}})}{\partial\overline{\mathbf{C}}}\overline{\mathbf{F}}^{\mathrm{T}} + \mathbf{F}\frac{\partial\Psi}{\partial\mathbf{C}}\mathbf{F}^{\mathrm{T}}\right],$$
$$\hat{\mathbf{t}}^4 = m\left[2\overline{\mathbf{F}}\frac{\partial\Sigma^*(\overline{\mathbf{C}})}{\partial\overline{\mathbf{C}}}\mathbf{V}^1 + \mathbf{F}^{-\mathrm{T}}\frac{\partial\Psi}{\partial\boldsymbol{\beta}_1}\right], \tag{53}$$
$$\hat{\mathbf{t}}^5 = m\left[2\overline{\mathbf{F}}\frac{\partial\Sigma^*(\overline{\mathbf{C}})}{\partial\overline{\mathbf{C}}}\mathbf{V}^2 + \mathbf{F}^{-\mathrm{T}}\frac{\partial\Psi}{\partial\boldsymbol{\beta}_2}\right],$$

where the remainder of $\hat{\mathbf{t}}^i$ are determined by Eq. (37).

At present, it is not known how to develop a specific form for the inhomogeneous strain energy Ψ for general cases. However, as a first approximation, it is reasonable to assume that Ψ is a quadratic function of the strains $\boldsymbol{\beta}_\alpha$ and the strain $\mathbf{E}$ defined by

$$\mathbf{E} = \tfrac{1}{2}(\mathbf{C} - \mathbf{I}). \tag{54}$$

Then, since Ψ must satisfy restrictions (52), it can be shown that Ψ cannot depend on the strain $\mathbf{E}$. Consequently, the quadratic function takes the form

$$2m\Psi = \mathbf{K}^{\alpha\beta}\cdot(\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta), \tag{55}$$

where $\mathbf{K}^{\alpha\beta}$ are four constant second-order tensors that satisfy the restrictions

$$(\mathbf{K}^{\beta\alpha})^{\mathrm{T}} = \mathbf{K}^{\alpha\beta}. \tag{56}$$

These symmetries indicate that the tensors $\mathbf{K}^{\alpha\beta}$ have 21 independent components. Moreover, using this form it follows that

$$m\frac{\partial\Psi}{\partial\boldsymbol{\beta}_\alpha} = \mathbf{K}^{\alpha\beta}\boldsymbol{\beta}_\beta. \tag{57}$$

As a special case, it is of interest to consider a three-dimensionally isotropic material and use the work of Flory (1961), which defines a nonlinear pure measure of distortion. Then as a simple case, the three-dimensional strain energy function can be expressed in the form of a generalized compressible Mooney–Rivlin material such that

$$2\rho_0^*\Sigma^*(\overline{\mathbf{C}}) = 2K^*[\overline{J} - 1 - \ln(\overline{J})] + \mu^*(\overline{\mathbf{C}}'\cdot\mathbf{I} - 3), \tag{58}$$

where K^* is a material constant related to the bulk modulus, μ^* is a material constant related to the shear modulus, and the pure measures of distortional deformation are defined by the unimodular tensors $\overline{\mathbf{F}}'$, $\overline{\mathbf{C}}'$ and $\overline{\mathbf{B}}'$

$$\overline{\mathbf{F}}' = \overline{J}^{-1/3}\overline{\mathbf{F}},\quad \overline{J} = \det\overline{\mathbf{F}},\quad \overline{\mathbf{C}}' = \overline{\mathbf{F}}'^{\mathrm{T}}\overline{\mathbf{F}}',\quad \overline{\mathbf{B}}' = \overline{\mathbf{F}}'\overline{\mathbf{F}}'^{\mathrm{T}}. \tag{59}$$

Moreover, it can be shown that

$$\frac{\partial\overline{J}}{\partial\overline{\mathbf{C}}} = \frac{1}{2}\overline{J}\overline{\mathbf{C}}^{-1},\quad \frac{\partial(\overline{\mathbf{C}}'\cdot\mathbf{I})}{\partial\overline{\mathbf{C}}} = \overline{J}^{-2/3}\left[\mathbf{I} - \frac{1}{3}(\overline{\mathbf{C}}\cdot\mathbf{I})\overline{\mathbf{C}}^{-1}\right],$$
$$2m\frac{\partial\Sigma^*(\overline{\mathbf{C}})}{\partial\overline{\mathbf{C}}} = D^{1/2}VK^*[\overline{J} - 1]\overline{\mathbf{C}}^{-1} + D^{1/2}V\mu^*\overline{J}^{-2/3}\left[\mathbf{I} - \frac{1}{3}(\overline{\mathbf{C}}\cdot\mathbf{I})\overline{\mathbf{C}}^{-1}\right]. \tag{60}$$

In general, it is possible to define normalized measures κ^i_α of inhomogeneous strains by the expressions

$$\kappa^i_\alpha = L\mathbf{D}^i\cdot\boldsymbol{\beta}_\alpha\quad \text{for } i = 1, 2, 3, \tag{61}$$

where L is the distance between the centroids of the cross-sections ∂P_α of the Cosserat in its reference configuration. Then, taking Ψ to be a nonlinear function of these variables

$$\Psi = \hat{\Psi}(\kappa^i_\alpha), \tag{62}$$

it follows that

$$\frac{\partial \Psi}{\partial \mathbf{C}} = 0, \qquad \frac{\partial \Psi}{\partial \boldsymbol{\beta}_\alpha} = L\sum_{i=1}^{3}\left[\frac{\partial \hat{\Psi}}{\partial \kappa^i_\alpha}\mathbf{D}^i\right]. \tag{63}$$

Thus, with the help of Eqs. (60) and (63), the constitutive equations (53) reduce to

$$\begin{aligned}
d^{1/2}\hat{\mathbf{T}} &= D^{1/2}VK^*\left[\overline{J}-1\right]\mathbf{I} + V\mu^*\left[\overline{\mathbf{B}}' - \frac{1}{3}(\overline{\mathbf{B}}'\cdot\mathbf{I})\mathbf{I}\right],\\
\hat{\mathbf{t}}^4 &= D^{1/2}VK^*[\overline{J}-1]\overline{\mathbf{F}}^{-\mathrm{T}}\mathbf{V}^1 + D^{1/2}V\mu^*\overline{J}^{-2/3}\left[\overline{\mathbf{F}} - \frac{1}{3}(\overline{\mathbf{C}}\cdot\mathbf{I})\overline{\mathbf{F}}^{-\mathrm{T}}\right]\mathbf{V}^1 + mL\sum_{i=1}^{3}\left[\frac{\partial\hat{\Psi}}{\partial\kappa^i_1}\mathbf{d}^i\right],\\
\hat{\mathbf{t}}^5 &= D^{1/2}VK^*[\overline{J}-1]\overline{\mathbf{F}}^{-\mathrm{T}}\mathbf{V}^2 + D^{1/2}V\mu^*\overline{J}^{-2/3}\left[\overline{\mathbf{F}} - \frac{1}{3}(\overline{\mathbf{C}}\cdot\mathbf{I})\overline{\mathbf{F}}^{-\mathrm{T}}\right]\mathbf{V}^2 + mL\sum_{i=1}^{3}\left[\frac{\partial\hat{\Psi}}{\partial\kappa^i_2}\mathbf{d}^i\right].
\end{aligned} \tag{64}$$

For a general quadratic function of the six variables κ^i_α, it is necessary to specify 21 material constants. However, as a special simple case, Ψ can be specified in the form

$$2m\Psi = D^{1/2}V[k_1(\kappa^3_1)^2 + k_2(\kappa^3_2)^2 + k_3(\omega_1)^2 + k_4(\kappa^1_1)^2 + k_5(\kappa^2_2)^2 + k_6(\omega_2)^2 + k_7(\kappa^1_1\kappa^2_2) + k_8(\omega_1\omega_2)], \tag{65a}$$

$$\omega_1 = \frac{1}{2}(\kappa^2_1 - \kappa^1_2), \qquad \omega_2 = \frac{1}{2}(\kappa^2_1 + \kappa^1_2), \tag{65b, c}$$

where k_i are material constants and the variables ω_α have been introduced for convenience. It will be shown that (k_1, k_2) control bending, k_3 controls torsion, (k_4, k_5) control hour glassing due to extension of the cross-section, and k_6 controls hour glassing due to shearing of the cross-section of the rod element. The constants k_7 and k_8 are introduced for later comparison with the expression for the strain energy associated with the Galerkin approach (Appendix D), but they are set equal to zero for the Cosserat model

$$k_7 = k_8 = 0. \tag{66}$$

If the constants k_1–k_6 are positive and Eq. (66) holds, then it can be easily seen that function (65) is a positive definite function of the variables κ^i_α. Moreover, it is noted that this expression is consistent with form (55) when $\mathbf{K}^{\alpha\beta}$ are specified by

$$\begin{aligned}
\mathbf{K}^{11} &= D^{1/2}VL^2\left[k_4(\mathbf{D}^1\otimes\mathbf{D}^1) + \tfrac{1}{4}(k_3+k_6)(\mathbf{D}^2\otimes\mathbf{D}^2) + k_1(\mathbf{D}^3\otimes\mathbf{D}^3)\right],\\
\mathbf{K}^{12} &= D^{1/2}VL^2\left[\tfrac{1}{4}(-k_3+k_6)(\mathbf{D}^2\otimes\mathbf{D}^1)\right], \qquad \mathbf{K}^{21} = D^{1/2}VL^2\left[\tfrac{1}{4}(-k_3+k_6)(\mathbf{D}^1\otimes\mathbf{D}^2)\right],\\
\mathbf{K}^{22} &= D^{1/2}VL^2\left[\tfrac{1}{4}(k_3+k_6)(\mathbf{D}^1\otimes\mathbf{D}^1) + k_5(\mathbf{D}^2\otimes\mathbf{D}^2) + k_2(\mathbf{D}^3\otimes\mathbf{D}^3)\right].
\end{aligned} \tag{67}$$

6. Reformulation of the balance laws

Since the theory of a Cosserat point, discussed in the previous sections, has been developed by a direct approach, the kinematic quantities $\mathbf{D}_i$ and $\mathbf{d}_i$ are not necessarily connected to the kinematic assumptions (A.1) and (A.4) associated with the derivation from three-dimensions discussed in Appendix A. However, for the numerical procedures that will be developed later, it is necessary to connect neighboring Cosserat points by kinematic and kinetic conditions at their common cross-sectional ends. In this regard (Fig. 2), let the end ∂P_1, be characterized by the two vectors $\mathbf{d}_{\alpha 1}(t)$ and let its centroid be located by the position vector $\mathbf{d}_{01}(t)$. Also, let the end ∂P_2 be characterized by the two vectors $\mathbf{d}_{\alpha 2}(t)$ and let its centroid be located by the position vector $\mathbf{d}_{02}(t)$. Moreover, let the reference values of these vectors be denoted by $\mathbf{D}_{\alpha 1}, \mathbf{D}_{01}, \mathbf{D}_{\alpha 2}, \mathbf{D}_{02}$.

The kinematic assumption (A.1) suggests that the vectors $\mathbf{D}_i$ are given by

$$\begin{aligned} \mathbf{D}_0 &= \frac{1}{2}(\mathbf{D}_{01} + \mathbf{D}_{02}), \qquad \mathbf{D}_3 = \frac{1}{L}(\mathbf{D}_{02} - \mathbf{D}_{01}), \\ \mathbf{D}_1 &= \frac{1}{2}(\mathbf{D}_{11} + \mathbf{D}_{12}), \qquad \mathbf{D}_4 = \frac{1}{L}(\mathbf{D}_{12} - \mathbf{D}_{11}), \\ \mathbf{D}_2 &= \frac{1}{2}(\mathbf{D}_{21} + \mathbf{D}_{22}), \qquad \mathbf{D}_5 = \frac{1}{L}(\mathbf{D}_{22} - \mathbf{D}_{21}), \end{aligned} \tag{68}$$

where L is the distance between the centroids of the ends ∂P_α in the reference configuration defined in Eq. (B.8). Similarly, the kinematic assumption (A.4) suggests that the vectors $\mathbf{d}_i$ are given by

$$\begin{aligned} \mathbf{d}_0 &= \frac{1}{2}(\mathbf{d}_{01} + \mathbf{d}_{02}), \qquad \mathbf{d}_3 = \frac{1}{L}(\mathbf{d}_{02} - \mathbf{d}_{01}), \\ \mathbf{d}_1 &= \frac{1}{2}(\mathbf{d}_{11} + \mathbf{d}_{12}), \qquad \mathbf{d}_2 = \frac{1}{2}(\mathbf{d}_{21} + \mathbf{d}_{22}), \\ \mathbf{d}_4 &= \frac{1}{L}(\mathbf{d}_{12} - \mathbf{d}_{11}), \qquad \mathbf{d}_5 = \frac{1}{L}(\mathbf{d}_{22} - \mathbf{d}_{21}). \end{aligned} \tag{69}$$

Also, using these expressions, the director velocities $\mathbf{w}_i$ can be written in the forms

$$\begin{aligned} \mathbf{w}_0 &= \frac{1}{2}(\dot{\mathbf{d}}_{01} + \dot{\mathbf{d}}_{02}), \qquad \mathbf{w}_3 = \frac{1}{L}(\dot{\mathbf{d}}_{02} - \dot{\mathbf{d}}_{01}), \\ \mathbf{w}_1 &= \frac{1}{2}(\dot{\mathbf{d}}_{11} + \dot{\mathbf{d}}_{12}), \qquad \mathbf{w}_2 = \frac{1}{2}(\dot{\mathbf{d}}_{21} + \dot{\mathbf{d}}_{22}), \\ \mathbf{w}_4 &= \frac{1}{L}(\dot{\mathbf{d}}_{12} - \dot{\mathbf{d}}_{11}), \qquad \mathbf{w}_5 = \frac{1}{L}(\dot{\mathbf{d}}_{22} - \dot{\mathbf{d}}_{21}). \end{aligned} \tag{70}$$

The constitutive equations discussed in Section 5 are valid for general nonlinear deformations and they produce exact results for all homogeneous deformations. The specification, Eq. (68), is valid in general and specification (69) is consistent with exact results for all nonlinear homogeneous deformations. Moreover, Eq. (69) is valid for large displacements and rotations. Also, it will be shown that Eq. (69) produces reasonably accurate results if the inhomogeneous deformation is relatively small. However, large overall inhomogeneous deformation of the rod can be described if the rod is modeled using a sufficiently large number of elements.

From the kinetical point of view, it is noted that the assigned fields $\mathbf{b}^i$ in the momentum equation (5b) separate into two parts Eq. (6b): one part $\mathbf{B}^i$ due to body force and surface tractions on the lateral surface ∂P_{L} of the rod section, and the other part $(\mathbf{m}_1^i + \mathbf{m}_2^i)$ due to surface tractions on the ends ∂P_α of the rod section. Moreover, in view of the definitions in Appendix B, the director couples $(\mathbf{m}_\alpha^3, \mathbf{m}_\alpha^4, \mathbf{m}_\alpha^5)$ are related to the quantities $(\mathbf{m}_\alpha^1, \mathbf{m}_\alpha^2, \mathbf{m}_\alpha^3)$ by formulas (B.8). This means that the balance laws Eq. (5b) can be solved for the vectors $(\mathbf{m}_\alpha^1, \mathbf{m}_\alpha^2, \mathbf{m}_\alpha^3)$ and rewritten in the forms

$$
\begin{aligned}
\mathbf{m}_1^0 &= \frac{1}{L}\left[\frac{L}{2}\left\{\sum_{j=0}^{5} my^{0j}\dot{\mathbf{w}}_j - m\mathbf{B}^0\right\} - \left\{\sum_{j=0}^{5} my^{3j}\dot{\mathbf{w}}_j - m\mathbf{B}^3 + \mathbf{t}^3\right\}\right],\\
\mathbf{m}_2^0 &= \frac{1}{L}\left[\frac{L}{2}\left\{\sum_{j=0}^{5} my^{0j}\dot{\mathbf{w}}_j - m\mathbf{B}^0\right\} + \left\{\sum_{j=0}^{5} my^{3j}\dot{\mathbf{w}}_j - m\mathbf{B}^3 + \mathbf{t}^3\right\}\right],\\
\mathbf{m}_1^1 &= \frac{1}{L}\left[\frac{L}{2}\left\{\sum_{j=0}^{5} my^{1j}\dot{\mathbf{w}}_j - m\mathbf{B}^1 + \mathbf{t}^1\right\} - \left\{\sum_{j=0}^{5} my^{4j}\dot{\mathbf{w}}_j - m\mathbf{B}^4 + \mathbf{t}^4\right\}\right],\\
\mathbf{m}_2^1 &= \frac{1}{L}\left[\frac{L}{2}\left\{\sum_{j=0}^{5} my^{1j}\dot{\mathbf{w}}_j - m\mathbf{B}^1 + \mathbf{t}^1\right\} + \left\{\sum_{j=0}^{5} my^{4j}\dot{\mathbf{w}}_j - m\mathbf{B}^4 + \mathbf{t}^4\right\}\right],\\
\mathbf{m}_1^2 &= \frac{1}{L}\left[\frac{L}{2}\left\{\sum_{j=0}^{5} my^{2j}\dot{\mathbf{w}}_j - m\mathbf{B}^2 + \mathbf{t}^2\right\} - \left\{\sum_{j=0}^{5} my^{5j}\dot{\mathbf{w}}_j - m\mathbf{B}^5 + \mathbf{t}^5\right\}\right],\\
\mathbf{m}_2^2 &= \frac{1}{L}\left[\frac{L}{2}\left\{\sum_{j=0}^{5} my^{2j}\dot{\mathbf{w}}_j - m\mathbf{B}^2 + \mathbf{t}^2\right\} + \left\{\sum_{j=0}^{5} my^{5j}\dot{\mathbf{w}}_j - m\mathbf{B}^5 + \mathbf{t}^5\right\}\right].
\end{aligned}
\tag{71}
$$

For the special case of equilibrium ($\mathbf{w}_i = 0$), with no body force and no surface tractions applied to the lateral surface of the rod section ($\mathbf{B}^i = 0$), these equations simplify to

$$
\begin{aligned}
&\mathbf{m}_1^0 = -\frac{1}{L}\mathbf{t}^3, \qquad \mathbf{m}_2^0 = \frac{1}{L}\mathbf{t}^3,\\
&\mathbf{m}_1^1 = \frac{1}{2}\mathbf{t}^1 - \frac{1}{L}\mathbf{t}^4, \qquad \mathbf{m}_2^1 = \frac{1}{2}\mathbf{t}^1 + \frac{1}{L}\mathbf{t}^4,\\
&\mathbf{m}_1^2 = \frac{1}{2}\mathbf{t}^2 - \frac{1}{L}\mathbf{t}^5, \qquad \mathbf{m}_2^2 = \frac{1}{2}\mathbf{t}^2 + \frac{1}{L}\mathbf{t}^5.
\end{aligned}
\tag{72}
$$

7. Boundary conditions

From the point of view of the direct approach to the theory of a Cosserat point presented in Sections 2 through 5, the balance laws are written in terms of ordinary differential equations, which are functions of time only. Consequently, the solution of these equations requires initial conditions and not boundary conditions. However, the objective of this paper is to use a number of connected Cosserat points to model the motion of a rod, which requires boundary conditions on its ends. Moreover, in order to connect neighboring Cosserat points, it is necessary to discuss kinematical and kinetical conditions at the ends ∂P_α.

The nature of these boundary conditions are determined by examining expression (11b) for the rate of work done on the Cosserat point. Specifically, formulas (6b), (70) and (B.8) are used to rewrite Eq. (11b) in the form

$$
\mathscr{W} = \left[\sum_{i=0}^{5} m\mathbf{B}^i\cdot\mathbf{w}_i\right] + \left[\mathbf{m}_1^0\cdot\dot{\mathbf{d}}_{01} + \mathbf{m}_1^1\cdot\dot{\mathbf{d}}_{11} + \mathbf{m}_1^2\cdot\dot{\mathbf{d}}_{21}\right] + \left[\mathbf{m}_2^0\cdot\dot{\mathbf{d}}_{02} + \mathbf{m}_2^1\cdot\dot{\mathbf{d}}_{12} + \mathbf{m}_2^2\cdot\dot{\mathbf{d}}_{22}\right]. \tag{73}
$$

It then follows that the boundary conditions on the ends ∂P_α are determined by

$$
\begin{aligned}
&\text{specifying } \{\mathbf{d}_{01} \text{ or } \mathbf{m}_1^0\} \quad\text{and}\quad \{\mathbf{d}_{11} \text{ or } \mathbf{m}_1^1\} \quad\text{and}\quad \{\mathbf{d}_{21} \text{ or } \mathbf{m}_1^2\} \quad \text{on } \partial P_1,\\
&\text{specifying } \{\mathbf{d}_{02} \text{ or } \mathbf{m}_2^0\} \quad\text{and}\quad \{\mathbf{d}_{12} \text{ or } \mathbf{m}_2^1\} \quad\text{and}\quad \{\mathbf{d}_{22} \text{ or } \mathbf{m}_2^2\} \quad \text{on } \partial P_2.
\end{aligned}
\tag{74}
$$

Obviously, mixed conditions and mixed–mixed conditions can be formulated where some components of the kinematic quantities and other components of the work conjugate kinetic quantities are specified.

Also, the moments $\mathbf{m}_\alpha$ applied to the ends ∂P_α about their centroids are determined by Eq. (B.9) and can be expressed in the forms

$$\begin{aligned}\mathbf{m}_1 &= \left(\mathbf{d}_1 - \frac{L}{2}\mathbf{d}_4\right) \times \mathbf{m}_1^1 + \left(\mathbf{d}_2 - \frac{L}{2}\mathbf{d}_5\right) \times \mathbf{m}_1^2 \quad \text{on } \partial P_1,\\ \mathbf{m}_2 &= \left(\mathbf{d}_1 + \frac{L}{2}\mathbf{d}_4\right) \times \mathbf{m}_2^1 + \left(\mathbf{d}_2 + \frac{L}{2}\mathbf{d}_5\right) \times \mathbf{m}_2^2 \quad \text{on } \partial P_2.\end{aligned} \tag{75}$$

Thus, in addition to specifying the forces $\mathbf{m}_\alpha^0$, the boundary conditions (74) require specification of the couples $\mathbf{m}_\alpha^1$ and $\mathbf{m}_\alpha^2$, which contain more information than the moment $\mathbf{m}_\alpha$.

8. Determination of constitutive coefficients for beams

In this section, attention is confined to the simple case where the rod, in its reference configuration, is a straight beam with a constant rectangular cross-section. With respect to the fixed orthonormal base vectors $\mathbf{e}_i$ of a rectangular Cartesian coordinate system, the vectors $\mathbf{D}_{01}, \mathbf{D}_{\alpha 1}, \mathbf{D}_{02}, \mathbf{D}_{\alpha 2}$ are given by

$$\mathbf{D}_{01} = 0, \quad \mathbf{D}_{\alpha 1} = \mathbf{e}_\alpha, \quad \mathbf{D}_{02} = L\mathbf{e}_3, \quad \mathbf{D}_{\alpha 2} = \mathbf{e}_\alpha, \tag{76}$$

where L is the length of the beam. Also, the three-dimensional region of space occupied by the beam in its reference configuration is characterized by

$$|\theta^1| \leqslant \frac{H}{2}, \qquad |\theta^2| \leqslant \frac{W}{2}, \qquad 0 \leqslant \theta^3 \leqslant L, \tag{77}$$

where θ^i are convected coordinates, H is the height and W is the width of the cross-section. It then follows from Eq. (68) that the directors $\mathbf{D}_i$ are given by

$$\mathbf{D}_0 = \frac{L}{2}\mathbf{e}_3, \quad \mathbf{D}_\alpha = \mathbf{e}_\alpha, \quad \mathbf{D}_3 = \mathbf{e}_3, \quad \mathbf{D}_4 = 0, \quad \mathbf{D}_5 = 0, \tag{78}$$

so the directors $\mathbf{D}_\alpha$ align with the principal directions $\mathbf{e}_\alpha$ of the cross-section.

Moreover, the material is assumed to be elastically isotropic, with the strain energy function being given by Eqs. (50), (58), (65) and (66), and attention is confined to equilibrium so that the effects of dissipation (41) vanish. For this geometry, it follows that $D^{1/2}$, V in Eq. (C.9), and the vectors $\mathbf{V}^\alpha$ in Eq. (C.12), are given by

$$D^{1/2} = 1, \qquad V = LHW, \qquad \mathbf{V}^\alpha = 0, \tag{79}$$

so constitutive equation (64) reduce to

$$\begin{aligned}\mathbf{T} &= VK^*[J-1]\mathbf{I} + J^{-2/3}V\mu^*\left[\mathbf{B} - \frac{1}{3}(\mathbf{B}\cdot\mathbf{I})\mathbf{I}\right],\\ \mathbf{t}^4 &= VL\left[k_4(\kappa_1^1)\mathbf{d}^1 + \tfrac{1}{2}(k_3\omega_1 + k_6\omega_2)\mathbf{d}^2 + k_1(\kappa_1^3)\mathbf{d}^3\right],\\ \mathbf{t}^5 &= VL\left[\tfrac{1}{2}(-k_3\omega_1 + k_6\omega_2)\mathbf{d}^1 + k_5(\kappa_2^2)\mathbf{d}^2 + k_2(\kappa_2^3)\mathbf{d}^3\right].\end{aligned} \tag{80}$$

In order to investigate the nonlinearity of these constitutive equations for uniaxial stress in the $\mathbf{e}_3$ direction, it is convenient to specify

$$\mathbf{d}_{01} = 0, \qquad \mathbf{d}_{02} = \lambda_3 L\mathbf{e}_3, \mathbf{d}_{11} = \lambda_1\mathbf{e}_1, \quad \mathbf{d}_{12} = \lambda_1\mathbf{e}_1, \quad \mathbf{d}_{21} = \lambda_2\mathbf{e}_2, \quad \mathbf{d}_{22} = \lambda_2\mathbf{e}_2, \tag{81}$$

where λ_1 and λ_2 determine the stretches of the cross-section and λ_3 determines the axial stretch. It then follows from Eqs. (16), (18), (61) and (69), that

$$\begin{aligned}
&\mathbf{d}_0 = \frac{1}{2}\lambda_3 L\mathbf{e}_3, \quad \mathbf{d}_1 = \lambda_1\mathbf{e}_1, \quad \mathbf{d}_2 = \lambda_2\mathbf{e}_2, \quad \mathbf{d}_3 = \lambda_3\mathbf{e}_3, \quad \mathbf{d}_4 = 0, \quad \mathbf{d}_5 = 0, \\
&\mathbf{F} = \lambda_1(\mathbf{e}_1 \otimes \mathbf{e}_1) + \lambda_2(\mathbf{e}_2 \otimes \mathbf{e}_2) + \lambda_3(\mathbf{e}_3 \otimes \mathbf{e}_3), \qquad J = \lambda_1\lambda_2\lambda_3, \\
&\kappa_1^1 = \kappa_2^2 = \kappa_1^3 = \kappa_2^3 = \omega_1 = \omega_2 = 0.
\end{aligned} \tag{82}$$

Thus, constitutive equation (80) become

$$\begin{aligned}
\mathbf{T} = VK^*[J-1]\mathbf{I} + \frac{1}{3}J^{-2/3}V\mu^*[\{2\lambda_1^2 - \lambda_2^2 - \lambda_3^2\}(\mathbf{e}_1 \otimes \mathbf{e}_1) \\
+ \{-\lambda_1^2 + 2\lambda_2^2 - \lambda_3^2\}(\mathbf{e}_2 \otimes \mathbf{e}_2) + \{-\lambda_1^2 - \lambda_2^2 + 2\lambda_3^2\}(\mathbf{e}_3 \otimes \mathbf{e}_3)], \quad \mathbf{t}^4 = \mathbf{t}^5 = 0.
\end{aligned} \tag{83}$$

Also, it is convenient to introduce the engineering strains ε_{i} and the volumetric strain ε_{v} by the formulas

$$\varepsilon_{\mathrm{i}} = \lambda_i - 1 \quad \text{for } i = 1,2,3, \qquad \varepsilon_{\mathrm{v}} = J - 1. \tag{84}$$

Now, for uniaxial stress in the $\mathbf{e}_3$ direction, the boundary conditions at the ends ∂P_α are specified by

$$\begin{aligned}
&\mathbf{m}_1^0 = -P, \quad \mathbf{m}_1^1 = 0, \quad \mathbf{m}_1^2 = 0, \quad \mathbf{m}_1 = 0, \\
&\mathbf{m}_2^0 = P, \quad \mathbf{m}_2^1 = 0, \quad \mathbf{m}_2^2 = 0, \quad \mathbf{m}_2 = 0,
\end{aligned} \tag{85}$$

where P is the force acting on the ends and use has been made of definition (75) for the moment $\mathbf{m}_\alpha$. It then follows that, in the absence of body forces and surface tractions on the lateral surface of the rod element, equilibrium equation (72) require

$$\mathbf{t}^1 = 0, \quad \mathbf{t}^2 = 0, \quad \mathbf{t}^3 = LP\mathbf{e}_3, \quad \mathbf{t}^4 = 0, \quad \mathbf{t}^5 = 0. \tag{86}$$

Thus, with the help of Eqs. (37) and (83), these equations yield

$$\begin{aligned}
&\lambda_1 = \lambda_2, \qquad K^*[J-1] + \frac{1}{3}J^{-2/3}\mu^*(\lambda_1^2 - \lambda_3^2) = 0, \\
&\frac{P}{A} = \frac{1}{\lambda_3}\left[K^*(J-1) + \frac{2}{3}J^{-2/3}\mu^*(\lambda_3^2 - \lambda_1^2)\right], \qquad A = \frac{V}{L} = HW,
\end{aligned} \tag{87}$$

where A is the reference cross-sectional area. Moreover, for homogeneous deformation, it follows from Eq. (B.7) that the three-dimensional Cauchy stress $\mathbf{T}^*$ is related to $\mathbf{T}$ by the expressions

$$\mathbf{T}^* = \frac{1}{JV}\mathbf{T}, \qquad T_{33}^* = \mathbf{T}^* \cdot (\mathbf{e}_3 \otimes \mathbf{e}_3) = \frac{\lambda_3 P}{JA} = \frac{P}{\lambda_1^2 A}. \tag{88}$$

By specifying λ_3 and solving part two of Eq. (87) for λ_1, it is possible to determine the response to uniaxial stress. For the specific numerical example to be discussed in the rest of this paper, the material constants are specified in terms of μ^* and Poisson's ratio v^* such that

$$\mu^* = 10 \text{ GPa}, \quad v^* = 0.25, \quad K^* = \frac{2\mu^*(1+v^*)}{3(1-2v^*)}, \quad E^* = 2\mu^*(1+v^*). \tag{89}$$

These values were chosen for simplicity and they do not correspond to any particular material.

Fig. 3(a) shows that the normalized Cauchy stress T_{33}^*/E^*A (true stress) and the normalized engineering stress (P/E^*A) are both nonlinear functions of the engineering axial strain ε_3. Also, Fig. 3(b) plots the

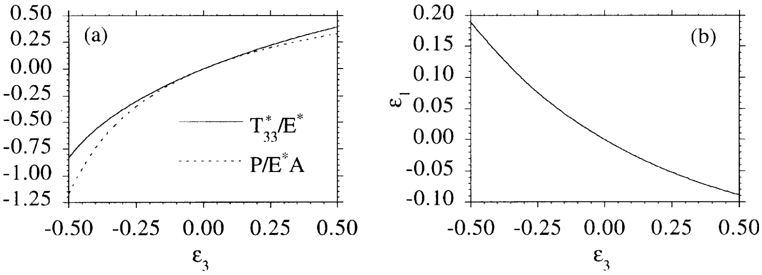

Fig. 3. Solution for uniaxial stress in the axial direction showing (a) the normalized Cauchy stress (T^*_{33}/E^*) and normalized engineering stress (P/E^*A) and (b) the lateral engineering strain ε_1 as functions of the axial engineering strain ε_3.

engineering lateral strain ε_1. In particular, notice from Fig. 3(a) that the change in cross-sectional area causes a softening effect for P during tension and a stiffening effect for P during compression.

It will presently be shown that the constants (k_1, k_2) can be determined by considering pure bending of the beam, and the constant k_3 can be determined by considering pure torsion of the beam. To this end, first consider the problem of pure bending when the beam is bent into a circular arc with $\mathbf{d}_{1\alpha}$ and $\mathbf{d}_{2\alpha}$ given by

$$\begin{aligned}
&\mathbf{d}_{01} = 0, \qquad \mathbf{d}_{02} = \lambda_3 L[\sin(\beta/2)\mathbf{e}_1 + \cos(\beta/2)\mathbf{e}_3], \\
&\mathbf{d}_{11} = \lambda_1 \mathbf{e}_1 = \left[\frac{\overline{\lambda}_1}{\cos(\beta/2)}\right]\mathbf{e}_1, \qquad \mathbf{d}_{12} = \lambda_1[\cos\beta\mathbf{e}_1 - \sin\beta\mathbf{e}_3], \\
&\mathbf{d}_{21} = \lambda_2\mathbf{e}_2, \qquad \mathbf{d}_{22} = \lambda_2\mathbf{e}_2,
\end{aligned} \tag{90}$$

where λ_1, λ_2, λ_3 are stretches to be determined, β is the angle defining the arc of the circle made by the centroid of the bent beam, and the quantity $\overline{\lambda}_1$ has been introduced for convenience. Now, using definition (69), it can be shown that

$$\begin{aligned}
&\mathbf{d}_0 = \frac{1}{2}\lambda_3 L\mathbf{e}'_3, \qquad \mathbf{d}_1 = \overline{\lambda}_1\mathbf{e}'_1, \qquad \mathbf{d}_2 = \lambda_2\mathbf{e}_2, \qquad \mathbf{d}_3 = \lambda_3\mathbf{e}'_3, \qquad \mathbf{d}_4 = -\frac{2\overline{\lambda}_1\tan(\beta/2)}{L}\mathbf{e}'_3, \qquad \mathbf{d}_5 = 0, \\
&\mathbf{e}'_1 = [\cos(\beta/2)\mathbf{e}_1 - \sin(\beta/2)\mathbf{e}_3], \qquad \mathbf{e}'_3 = [\sin(\beta/2)\mathbf{e}_1 + \cos(\beta/2)\mathbf{e}_3],
\end{aligned} \tag{91}$$

where the unit vectors $\mathbf{e}'_1$ and $\mathbf{e}'_3$ have been defined for convenience. Moreover, using these results, it follows from definitions (16), (18), (61) and (65) that

$$\begin{aligned}
&\mathbf{F} = \overline{\lambda}_1\mathbf{e}'_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3\mathbf{e}'_3 \otimes \mathbf{e}_3, \qquad J = \overline{\lambda}_1\lambda_2\lambda_3, \\
&\kappa^1_1 = \kappa^2_2 = \kappa^3_2 = 0, \qquad \omega_1 = \omega_2 = 0, \qquad \kappa^3_1 = -\frac{2\overline{\lambda}_1\tan(\beta/2)}{\lambda_3}.
\end{aligned} \tag{92}$$

Thus, constitutive equation (80) become

$$\begin{aligned}
\mathbf{T} = {} & VK^*[J-1]\mathbf{I} + \frac{1}{3}J^{-2/3}V\mu^*\left[2\overline{\lambda}_1^2 - \lambda_2^2 - \lambda_3^2\right](\mathbf{e}'_1 \otimes \mathbf{e}'_1) + \frac{1}{3}J^{-2/3}V\mu^*\left[-\overline{\lambda}_1^2 + 2\lambda_2^2 - \lambda_3^2\right](\mathbf{e}_2 \otimes \mathbf{e}_2) \\
& + \frac{1}{3}J^{-2/3}V\mu^*\left[-\overline{\lambda}_1^2 - \lambda_2^2 + 2\lambda_3^2\right](\mathbf{e}'_3 \otimes \mathbf{e}'_3), \\
\mathbf{t}^4 = {} & \left[\frac{VLk_1(\kappa^3_1)}{\lambda_3}\right]\mathbf{e}'_3, \qquad \mathbf{t}^5 = 0.
\end{aligned} \tag{93}$$

Now, for pure bending, the boundary conditions at the ends ∂P_α are specified by

$$\mathbf{m}_1^0 = 0, \quad \mathbf{m}_1^1 = \frac{M}{\bar{\lambda}_1}\mathbf{e}_3', \quad \mathbf{m}_1^2 = 0, \quad \mathbf{m}_1 = -M\mathbf{e}_2,$$
$$\mathbf{m}_2^0 = 0, \quad \mathbf{m}_2^1 = -\frac{M}{\bar{\lambda}_1}\mathbf{e}_3', \quad \mathbf{m}_2^2 = 0, \quad \mathbf{m}_2 = M\mathbf{e}_2, \tag{94}$$

where M is the magnitude of the moment and use has been made of definition (75) for the moment $\mathbf{m}_\alpha$. It then follows that, in the absence of body forces and surface tractions on the lateral surface of the rod element, equilibrium equation (72) require

$$\mathbf{t}^1 = 0, \quad \mathbf{t}^2 = 0, \quad \mathbf{t}^3 = 0, \quad \mathbf{t}^4 = -\frac{LM}{\bar{\lambda}_1}\mathbf{e}_3', \quad \mathbf{t}^5 = 0. \tag{95}$$

Thus, with the help of Eqs. (37), (93) and a specification of β, these equations yield four scalar equations for the four unknowns $(\bar{\lambda}_1, \lambda_2, \lambda_3, M)$

$$\bar{\lambda}_1 = \lambda_2, \qquad K^*[J-1] + \frac{1}{3}J^{-2/3}\mu^*[\bar{\lambda}_1^2 - \lambda_3^2] = 0,$$
$$K^*[J-1] - \frac{2}{3}J^{-2/3}\mu^*[\bar{\lambda}_1^2 - \lambda_3^2] = \frac{4k_1\bar{\lambda}_1^2\tan^2(\beta/2)}{\lambda_3^2},$$
$$M = \frac{2Vk_1\bar{\lambda}_1^2\tan(\beta/2)}{\lambda_3^2}. \tag{96}$$

Next, assuming that the centroids of the ends ∂P_α lie on a circle of radius r it can be shown that

$$\sin(\beta/2) = \frac{L\lambda_3}{2r}, \qquad M = \left[\frac{VLk_1\bar{\lambda}_1^2}{\lambda_3\cos(\beta/2)}\right]\frac{1}{r}. \tag{97}$$

Thus, neglecting quadratic terms in β, the linearized solution of Eq. (96) yields

$$\bar{\lambda}_1 = \lambda_1 = \lambda_2 = \lambda_3 = 1, \qquad \beta = \frac{L}{r}, \qquad M[Vk_1]\frac{L}{r} = [Vk_1]\beta. \tag{98}$$

Now, a similar expression for the moment can be obtained using the linearized three-dimensional theory for pure bending of rectangular bar

$$M = \left[\frac{E^*H^3W}{12}\right]\frac{1}{r} = \left[\frac{E^*H^3W}{12L}\right]\beta, \tag{99}$$

where E^* is Young's modulus. This suggests that coefficients k_1 and k_2 are given by

$$k_1 = \frac{E^*H^2}{12L^2}, \qquad k_2 = \frac{E^*W^2}{12L^2}, \tag{100}$$

where k_2 is determined by analogy for bending in the $\mathbf{e}_2$–$\mathbf{e}_3$ plane. Moreover, neglecting quadratic terms in β, Eqs. (90) (second part), (98) and (100) yield

$$\mathbf{d}_{02} = \left[\frac{6ML^2}{E^*H^3W}\right]\mathbf{e}_1 + L\mathbf{e}_3, \tag{101}$$

which can be shown to reproduce the correct value of the vertical $(\mathbf{e}_1)$ displacement.

The numerical solution of general rod problems will be discussed in Section 9. However, it is possible to numerically solve the problem of pure bending of a beam of reference length L_0 by merely dividing the beam into N equal segments, each of which is subjected to pure bending. In particular, consider a beam which is bent into a complete circle. Modeling each of the elements as a Cosserat point and using nonlinear equations (96), it is possible to define the length L and angle β for each of these elements by the expressions

$$L = \frac{L_0}{N}, \qquad \beta = \frac{2\pi}{N}. \tag{102a, b}$$

Moreover, it is convenient to define the normalized moment $\overline{M}$ such that

$$\overline{M} = \frac{6L_0 M}{E^* H^3 W \pi} = \frac{N\overline{\lambda}_1^2 \tan(\pi/N)}{\pi\lambda_3^2}. \tag{103}$$

Next, with the help of Eqs. (92) (second part) and (102), Eq. (96) (second and third part) can be rewritten in the forms

$$\overline{\lambda}_1^2 = \frac{\lambda_3^2}{\lambda_3^3 - \left[\frac{E^* H^2}{9K^* L_0^2}\right] N^2 \tan^2(\pi/N)}, \qquad J - 1 = \left[\frac{J^{-2/3}\mu^*}{3K^*}\right](\lambda_3^2 - \overline{\lambda}_1^2). \tag{104}$$

Consequently, these equations can be solved by specifying a value for N, guessing a value for λ_3, then calculating $\overline{\lambda}_1$ and iterating on the guess for λ_3 until Eq. (104b) is satisfied. Moreover, in determining the geometry of the beam, it is convenient to estimate the radius r of the middle line of the deformed beam by assuming that this middle line does not extend so that

$$\frac{r}{H} \approx \frac{L_0}{2\pi H}. \tag{105}$$

Thus, for $L_0/H = \pi$, the beam would be bent into a solid cylinder.

Fig. 4 shows the convergence of quantity (103) for different values of L_0/H. In particular, notice that for $L_0/H = 100$, the beam is quite thin and the normalized moment $\overline{M}$ converges to the value unity, which is consistent with the moment that would be predicted by describing each section of the beam with the linearized theory. Whereas, for thicker beams, the moment converges to a value less than that predicted by the linearized theory. This weaker response to bending seems to be caused by the effects of axial extension ($\varepsilon_3 > 0$) and cross-sectional contraction ($\varepsilon_1 < 0, \varepsilon_2 < 0$). Also, notice that the convergence is fairly rapid so that not many elements are needed to accurately predict the solution of this large deformation problem.

In order to determine the value of k_3, it is convenient to consider the problem of pure torsion of a straight beam about its axis. For this problem, the kinematics are specified by Eqs. (76)–(78) and

$$\begin{aligned}
&\mathbf{d}_{01} = 0, \qquad \mathbf{d}_{02} = \lambda_3 L \mathbf{e}_3,\\
&\mathbf{d}_{11} = \lambda_1 \mathbf{e}_1 = \left[\frac{\overline{\lambda}_1}{\cos(\gamma/2)}\right]\mathbf{e}_1, \qquad \mathbf{d}_{12} = \lambda_1[\cos\gamma \mathbf{e}_1 + \sin\gamma \mathbf{e}_2],\\
&\mathbf{d}_{21} = \lambda_2 \mathbf{e}_2 = \left[\frac{\overline{\lambda}_2}{\cos(\gamma/2)}\right]\mathbf{e}_2, \qquad \mathbf{d}_{22} = \lambda_2[-\sin\gamma \mathbf{e}_1 + \cos\gamma \mathbf{e}_2],
\end{aligned} \tag{106}$$

where λ_1, λ_2, λ_3 are stretches to be determined, γ is the angle defining the twist of the beam and the quantities $\overline{\lambda}_1$, $\overline{\lambda}_2$ have been introduced for convenience. Now, using definition (69), it can be shown that

$$\begin{aligned}
&\mathbf{d}_0 = \frac{1}{2}\lambda_3 L \mathbf{e}_3, \qquad \mathbf{d}_1 = \overline{\lambda}_1 \mathbf{e}_1', \qquad \mathbf{d}_2 = \overline{\lambda}_2 \mathbf{e}_2', \qquad \mathbf{d}_3 = \lambda_3 \mathbf{e}_3, \qquad \mathbf{d}_4 = \frac{2\overline{\lambda}_1 \tan(\gamma/2)}{L}\mathbf{e}_2',\\
&\mathbf{d}_5 = -\frac{2\overline{\lambda}_2 \tan(\gamma/2)}{L}\mathbf{e}_1', \qquad \mathbf{e}_1' = [\cos(\gamma/2)\mathbf{e}_1 + \sin(\gamma/2)\mathbf{e}_2], \qquad \mathbf{e}_2' = [-\sin(\gamma/2)\mathbf{e}_1 + \cos(\gamma/2)\mathbf{e}_2],
\end{aligned} \tag{107}$$

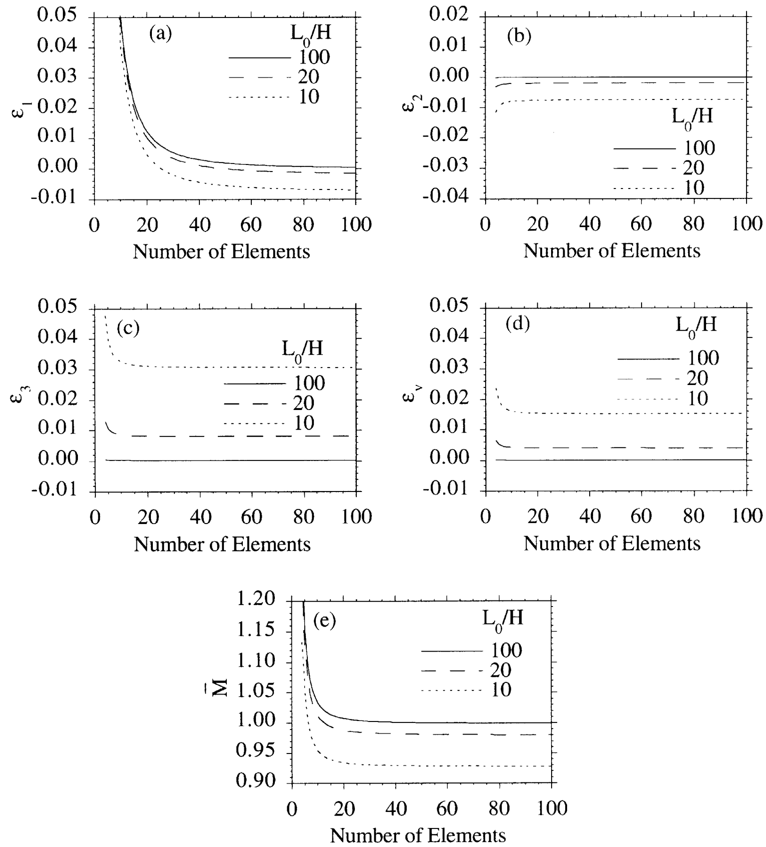

Fig. 4. Bending of a beam of reference length L_0 and thickness H into a circular ring showing the strains ε_1,ε_2,ε_3,ε_v and the normalized moment $\overline{M}$.

where the unit vectors $\mathbf{e}_1'$ and $\mathbf{e}_2'$ have been defined for convenience. Moreover, using these results, it follows from definitions (16), (18), (61) and (65) that

$$\mathbf{F} = \overline{\lambda}_1 \mathbf{e}_1' \otimes \mathbf{e}_1 + \overline{\lambda}_2 \mathbf{e}_2' \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3, \qquad J = \overline{\lambda}_1 \overline{\lambda}_2 \lambda_3, \qquad \kappa_1^1 = \kappa_2^2 = \kappa_1^3 = \kappa_2^3 = 0,$$
$$\omega_1 = \left[\frac{\overline{\lambda}_1}{\overline{\lambda}_2} + \frac{\overline{\lambda}_2}{\overline{\lambda}_1}\right] \tan(\gamma/2), \qquad \omega_2 = \left[\frac{\overline{\lambda}_1}{\overline{\lambda}_2} - \frac{\overline{\lambda}_2}{\overline{\lambda}_1}\right] \tan(\gamma/2). \tag{108a, b}$$

Thus, constitutive equation (80) become

$$\mathbf{T} = VK^*[J-1]\mathbf{I} + \frac{1}{3}J^{-2/3}V\mu^*\left[2\overline{\lambda}_1^2 - \overline{\lambda}_2^2 - \lambda_3^2\right](\mathbf{e}_1' \otimes \mathbf{e}_1') + \frac{1}{3}J^{-2/3}V\mu^*\left[-\overline{\lambda}_1^2 + 2\overline{\lambda}_2^2 - \lambda_3^2\right](\mathbf{e}_2' \otimes \mathbf{e}_2')$$
$$+ \frac{1}{3}J^{-2/3}V\mu^*\left[-\overline{\lambda}_1^2 - \overline{\lambda}_2^2 + 2\lambda_3^2\right](\mathbf{e}_3 \otimes \mathbf{e}_3), \tag{109}$$
$$\mathbf{t}^4 = \left[\frac{VL(k_3\omega_1 + k_6\omega_2)}{2\overline{\lambda}_2}\right]\mathbf{e}_2', \qquad \mathbf{t}^5 = \left[\frac{VL(-k_3\omega_1 + k_6\omega_2)}{2\overline{\lambda}_1}\right]\mathbf{e}_1'.$$

Now, for pure torsion, the boundary conditions at the ends ∂P_α are specified by

$$\mathbf{m}_1^0 = 0, \qquad \mathbf{m}_1^1 = -\frac{T}{2\bar{\lambda}_1}\mathbf{e}_2', \qquad \mathbf{m}_1^2 = \frac{T}{2\bar{\lambda}_2}\mathbf{e}_1', \qquad \mathbf{m}_1 = -T\mathbf{e}_3,$$
$$\mathbf{m}_2^0 = 0, \qquad \mathbf{m}_2^1 = \frac{T}{2\bar{\lambda}_1}\mathbf{e}_2', \qquad \mathbf{m}_2^2 = -\frac{T}{2\bar{\lambda}_2}\mathbf{e}_1', \qquad \mathbf{m}_2 = T\mathbf{e}_3, \tag{110}$$

where T is the magnitude of the torque, and use has been made of definition (75) for the moment $\mathbf{m}_\alpha$. It then follows that, in the absence of body forces and surface tractions on the lateral surface of the rod element, equilibrium equation (72) require

$$\mathbf{t}^1 = 0, \quad \mathbf{t}^2 = 0, \quad \mathbf{t}^3 = 0, \quad \mathbf{t}^4 = \frac{LT}{2\bar{\lambda}_1}\mathbf{e}_2', \quad \mathbf{t}^5 = -\frac{LT}{2\bar{\lambda}_2}\mathbf{e}_1'. \tag{111}$$

Thus, with the help of Eqs. (37), (109) and a specification of γ, these equations yield four scalar equations for the four unknowns $(\bar{\lambda}_1, \bar{\lambda}_2, \lambda_3, T)$

$$\bar{\lambda}_1 = \bar{\lambda}_2, \qquad K^*[J-1] + \frac{1}{3}J^{-2/3}\mu^*\left[\bar{\lambda}_1^2 - \lambda_3^2\right] = 2k_3\tan^2(\gamma/2),$$
$$K^*[J-1] - \frac{2}{3}J^{-2/3}\mu^*\left[\bar{\lambda}_1^2 - \lambda_3^2\right] = 0, \qquad T = 2Vk_3\tan(\gamma/2). \tag{112}$$

Next, neglecting quadratic terms in γ, the linearized solution of Eq. (112) yields

$$\bar{\lambda}_1 = \bar{\lambda}_2 = \lambda_1 = \lambda_2 = \lambda_3 = 1, \qquad T = Vk_3\gamma. \tag{113}$$

Now, a similar expression for the moment can be obtained using the linearized three-dimensional theory for pure torsion of rectangular bar

$$T = \left[\frac{\mu^* I_\mathrm{p}}{L}\right]\gamma, \qquad I_\mathrm{p} = \frac{1}{3}H^2W^2g(\eta), \qquad \eta = \frac{H}{W},$$
$$g(\eta) = \left[1 - \frac{192\eta}{\pi^5}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\tanh\left\{\frac{(2n-1)\pi}{2\eta}\right\}\right], \tag{114}$$

where I_p is the effective polar moment of the rectangular cross-sectional area (Sokolnikoff, 1956). Physically, it is clear that there can be no preference in the mathematical expression for either of the directions in the cross-section so the function $g(\eta)$ must have the property that

$$g(\eta) = g(1/\eta). \tag{115}$$

This restriction can be proved by expanding the hyperbolic tangent function in terms of power series (Gladwell, 1998). Now, comparison of Eqs. (113) and (114) yields

$$k_3 = \frac{\mu^* I_\mathrm{p}}{VL} = \frac{\mu^* HW}{3L^2}g(\eta). \tag{116}$$

It is possible to numerically solve the problem of pure torsion of a beam of reference length L_0 by merely dividing the beam into N equal segments, each of which is subjected to pure torsion. In particular, consider a beam with square cross-section ($H = W$) for which its ends are twisted relative to each other by an angle of 2π. Modeling each of the elements as a Cosserat point and using nonlinear equation (112), it is possible to define the length L and angle γ for each of these elements by the expressions

$$L = \frac{L_0}{N}, \qquad \gamma = \frac{2\pi}{N}. \tag{117}$$

Moreover, the strains ε_i and ε_v can be defined by Eq. (84), and the normalized torque $\bar{T}$ can be defined by

$$\overline{T} = \frac{T}{\frac{2\mu^* H^4 g(1)\pi}{3L_0}} = \frac{N}{\pi}\tan\frac{\pi}{N}, \tag{118}$$

where use has been made of Eqs. (112), (114) and (116) with $H = W$. Next, Eqs. (112) (second and third part) can be rewritten in the forms

$$J = 1 + \left[\frac{4\mu^* H^2 g(1)}{9K^* L_0^2}\right] N^2 \tan^2\left\{\frac{\pi}{N}\right\},$$
$$\overline{\lambda}_1^2 = \lambda_3^2 + \left[\frac{2J^{2/3} H^2 g(1)}{3L_0^2}\right] N^2 \tan^2\left\{\frac{\pi}{N}\right\}. \tag{119a, b}$$

Also, multiplying Eq. (119b) by λ_3 and using Eqs. (108b) and (119a), it can be shown that Eq. (119b) yields the cubic equation

$$\lambda_3^3 + \lambda_3\left[\frac{2J^{2/3} H^2 g(1)}{3L_0^2}\right] N^2 \tan^2\left\{\frac{\pi}{N}\right\} - J = 0 \tag{120}$$

to determine λ_3. Thus, for specified values of L_0/H and N, Eqs. (119) and (120) can be solved for $\overline{\lambda}_1 = \overline{\lambda}_2, \lambda_3$ and J. Then, Eqs. (106) (third part) and (118) can be solved for λ_1, λ_2 and $\overline{T}$.

Fig. 5 shows the convergence of the strains (84) and the normalized torque $\overline{T}$ (118) for different values of L_0/H. In particular, notice that $\overline{T}$ is independent of the value of L_0/H and depends only on the number of elements and the relative twist between the ends of the beam. However, the values of the strains are influenced by the value of L_0/H.

The material constants (k_4,k_5,k_6) are needed to control hour glassing of the deformation. In particular, it is noted that the varying shape of the cross-section of a bar hanging under its own weight is similar to the hour glassing mode of deformation associated with cross-sectional extension. However, in contrast with that problem, where the stress is uniaxial in the direction of gravity, the response in Eq. (80) due to cross-sectional hour glassing ($\kappa_1^1 \neq 0$, $\kappa_2^2 \neq 0$) is related to variation of stresses acting on planes, whose normals

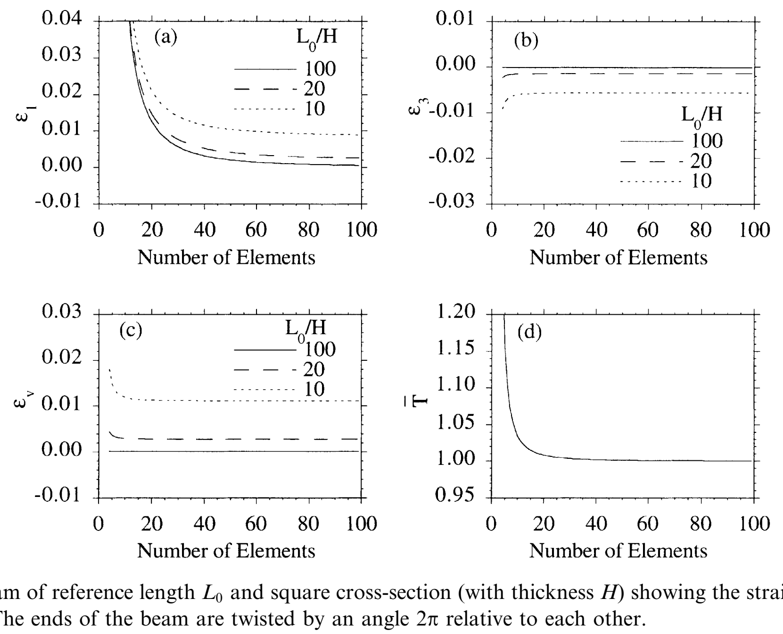

Fig. 5. Torsion of a beam of reference length L_0 and square cross-section (with thickness H) showing the strains $\varepsilon_1 = \varepsilon_2, \varepsilon_3, \varepsilon_v$ and the normalized torque $\overline{T}$. The ends of the beam are twisted by an angle 2π relative to each other.

are perpendicular to the direction of gravity (Eq. (B.5)). Consequently, comparison of the response (80) for this problem with the three-dimensional solution would indicate that k_4 and k_5 vanish. Therefore, at present, it seems reasonable to specify values for these constants that are large enough to control hour glassing and yet small enough to cause insignificant influence on the other modes of deformation. Thus, for simplicity, these constants are specified by

$$k_4 = k\mu^* \left[\frac{H^2}{L^2}\right], \qquad k_5 = k\mu^* \left[\frac{W^2}{L^2}\right], \qquad k_6 = k\mu^* \left[\frac{HW}{L^2}\right], \tag{121}$$

where k is a small constant ($k \approx 0.001$).

For comparison purposes, Appendix D outlines the usual Galerkin procedure for determining the response of a rod-like element. Specifically, a kinematic approximation is substituted into the three-dimensional strain energy and the result is integrated over the region of the element. This effectively determines the constitutive equations for the rod-like element, which are different from those developed above for the Cosserat theory. The reasons for these differences, and the need for special approximations like (D.11), are related to the use of the kinematic assumption pointwise in the cross-section (Rubin, 1996; O'Reilly, 1998; Zienkiewicz and Taylor, 1991, p. 161).

Next, attention is focused on the determination of the mass m and director inertia coefficients y^{ij} of the Cosserat point. For the simple case of the beam characterized by Eqs. (76) and (77), and with the assumption that the reference mass density ρ_0^* is constant, it can be shown that Eq. (B.1) yield

$$m = \rho_0^* V, \qquad V = HWL, \qquad y^{00} = 1, \tag{122}$$

$$\begin{aligned} &y^{11} = \frac{H^2}{12}, \qquad y^{22} = \frac{W^2}{12}, \qquad y^{33} = \frac{L^2}{12}, \\ &y^{44} = \frac{H^2L^2}{144}, \qquad y^{55} = \frac{W^2L^2}{144}, \qquad \text{all other } y^{ij} = 0. \end{aligned} \tag{123}$$

The values (123) are consistent with values that would be obtained by direct integration of Eq. (B.1). However, it has been shown in (Rubin, 1986) that the predictions of free vibrations of a rectangular parallelepiped can be improved if y^{11}, y^{22}, y^{33} are specified by

$$y^{11} = \frac{H^2}{\pi^2}, \qquad y^{22} = \frac{W^2}{\pi^2}, \qquad y^{33} = \frac{L^2}{\pi^2}. \tag{124}$$

In this regard, the director inertia coefficients represent information about vibrational mode shapes and do not merely characterize the distribution of mass in the Cosserat point.

To determine improved values for y^{44} and y^{55}, it is possible to consider the linearized problem of free bending vibrations of the beam. For this problem, it is convenient to neglect quadratic terms in the bending angle β and specify $\mathbf{d}_{i\alpha}$ by

$$\begin{aligned} &\mathbf{d}_{01} = 0, \qquad \mathbf{d}_{02} = L\mathbf{e}_3, \\ &\mathbf{d}_{11} = \mathbf{e}_1 + (\beta/2)\mathbf{e}_3, \qquad \mathbf{d}_{12} = \mathbf{e}_1 - (\beta/2)\mathbf{e}_3, \\ &\mathbf{d}_{21} = \mathbf{e}_2, \qquad \mathbf{d}_{22} = \mathbf{e}_2, \end{aligned} \tag{125}$$

so that Eq. (69) yield

$$\mathbf{d}_0 = \frac{L}{2}\mathbf{e}_3, \qquad \mathbf{d}_1 = \mathbf{e}_1, \qquad \mathbf{d}_2 = \mathbf{e}_2, \qquad \mathbf{d}_3 = \mathbf{e}_3, \qquad \mathbf{d}_4 = -\frac{\beta}{L}\mathbf{e}_3, \qquad \mathbf{d}_5 = 0. \tag{126}$$

Moreover, using these results, it follows from definitions (16), (18), (61) and (65), that

$$\mathbf{F} = \mathbf{I}, \quad J = 1, \quad \kappa_1^1 = \kappa_2^2 = \kappa_2^3 = 0, \quad \kappa_1^3 = -\beta, \quad \omega_1 = \omega_2 = 0. \tag{127}$$

Thus, with the help of Eq. (100), constitutive equation (80) become

$$\mathbf{T} = 0, \qquad \mathbf{t}^4 = -\left[\frac{E^*H^3W}{12}\right]\beta\mathbf{e}_3, \qquad \mathbf{t}^5 = 0. \tag{128}$$

For free vibrations the boundary conditions require

$$\mathbf{m}^0_\alpha = 0, \qquad \mathbf{m}^1_\alpha = 0, \qquad \mathbf{m}^2_\alpha = 0, \tag{129}$$

so that, in the absence of body forces and surface tractions on the lateral surface of the rod element, the equations of motion (5) yield a single scalar equation of the form

$$\left[\frac{my^{44}}{L}\right]\ddot{\beta} + \left[\frac{E^*H^3W}{12}\right]\beta = 0, \tag{130}$$

where use has been made of expression (70). It then follows that the natural frequency Ω for free–free symmetric bending vibration of the beam is given by

$$\Omega^2 = \left[\frac{E^*H^2}{12\rho_0^*}\right]\frac{1}{y^{44}}. \tag{131}$$

However, it can be shown (Graff, 1975, Section 3.2) that the frequency Ω^* of the lowest nontrivial free–free symmetric bending mode of a Bernoulli–Euler beam is given by

$$\Omega^{*2} = \left[\frac{E^*H^2}{12\rho_0^*}\right]\left[\frac{K}{L}\right]^4, \tag{132}$$

where K is the smallest nonzero positive solution of the equation

$$\cos(K)\cosh(K) = 1, \qquad K \approx 4.730 \approx (1.004)\left[\frac{3\pi}{2}\right]. \tag{133}$$

Thus, comparison of expressions (131) and (132) suggests that the director inertias y^{44} and y^{55} can be determined so that the single element predicts the correct lowest natural frequency for bending in either the $\mathbf{e}_1$–$\mathbf{e}_3$ plane or the $\mathbf{e}_2$–$\mathbf{e}_3$ plane by specifying

$$y^{44} = y^{55} = \left[\frac{L}{K}\right]^4 \approx \left[\frac{2L}{3\pi}\right]^4. \tag{134}$$

It is particularly interesting that the value of y^{44} is independent of H, the value of y^{55} is independent of W, and that these values are different from those of fourth and fifth part of Eq. (122) obtained by direct integration.

9. Numerical solution procedure for rod problems

As described in Section 1, the numerical solution procedure for rod problems is formulated by dividing the rod into N elements, each of which is modeled as a Cosserat point. Fig. 1 shows the kinematics of the Ith Cosserat point. With respect to the present configuration, the vector ${}_I\mathbf{d}_0^*$ locates the centroid of the Ith cross-section and the vectors ${}_I\mathbf{d}_1^*$ and ${}_I\mathbf{d}_2^*$ characterize material fibers in this cross-section. The values of these vectors in the reference configuration are given by ${}_I\mathbf{D}_0^*$, ${}_I\mathbf{D}_1^*$, ${}_I\mathbf{D}_2^*$.

In the previous sections, it was shown that an elastic rod-like Cosserat point is characterized by the kinematic and kinetic quantities

$$\{L, \mathbf{D}_i, \mathbf{d}_i, \mathbf{w}_i, m, y^{ij}, \Sigma, \Psi, \mathbf{t}^i, \mathbf{B}^i, \mathbf{m}_1^i, \mathbf{m}_2^i, \mathbf{m}_1, \mathbf{m}_2\}, \tag{135}$$

where the directors $\mathbf{D}_i$ and $\mathbf{d}_i$ are related to the kinematic quantities

$$\begin{aligned} &\{\mathbf{D}_{01}, \mathbf{D}_{11}, \mathbf{D}_{21}, \mathbf{D}_{02}, \mathbf{D}_{12}, \mathbf{D}_{22}\}, \\ &\{\mathbf{d}_{01}, \mathbf{d}_{11}, \mathbf{d}_{21}, \mathbf{d}_{02}, \mathbf{d}_{12}, \mathbf{d}_{22}\}, \end{aligned} \tag{136}$$

by Eqs. (68) and (69). Also, the director velocities $\mathbf{w}_i$ are given by Eq. (70). Here, it is convenient to identify the kinematic and kinetic quantities associated with the Ith Cosserat point using the same symbols as Eqs. (135) and (136), but with a subscript I. Thus, the Ith Cosserat point is characterized by the quantities

$$\begin{aligned} &\{{}_IL, {}_I\mathbf{D}_i, {}_I\mathbf{d}_i, {}_I\mathbf{w}_i, {}_Im, {}_Iy^{ij}, {}_I\Sigma, {}_I\Psi, {}_I\mathbf{t}^i, {}_I\mathbf{B}^i, {}_I\mathbf{m}_1^i, {}_I\mathbf{m}_2^i, {}_I\mathbf{m}_1, {}_I\mathbf{m}_2\}, \\ &\{{}_I\mathbf{D}_{01}, {}_I\mathbf{D}_{11}, {}_I\mathbf{D}_{21}, {}_I\mathbf{D}_{02}, {}_I\mathbf{D}_{12}, {}_I\mathbf{D}_{22}\}, \\ &\{{}_I\mathbf{d}_{01}, {}_I\mathbf{d}_{11}, {}_I\mathbf{d}_{21}, {}_I\mathbf{d}_{02}, {}_I\mathbf{d}_{12}, {}_I\mathbf{d}_{22}\}. \end{aligned} \tag{137}$$

For example, the vectors ${}_I\mathbf{D}_i$, ${}_I\mathbf{d}_i$, ${}_I\mathbf{w}_i$, ${}_I\mathbf{m}^i{}_1$ and ${}_I\mathbf{m}_2{}^i$ are given by formulas (68)–(71), with a subscript I placed on all quantities in those formulas.

The equations of motion for the Ith Cosserat point are coupled with those of its neighbors by specifying kinematic conditions at the common cross-sections of the forms

$$\begin{aligned} &{}_I\mathbf{d}_{01} = {}_I\mathbf{d}_0^*, \quad {}_I\mathbf{d}_{11} = {}_I\mathbf{d}_1^*, \quad {}_I\mathbf{d}_{21} = {}_I\mathbf{d}_2^*, \\ &{}_I\mathbf{d}_{02} = {}_{I+1}\mathbf{d}_0^*, \quad {}_I\mathbf{d}_{12} = {}_{I+1}\mathbf{d}_1^*, \quad {}_I\mathbf{d}_{22} = {}_{I+1}\mathbf{d}_2^* \quad \text{for } I = 1, 2, \ldots, N. \end{aligned} \tag{138}$$

Also, kinetic coupling is introduced by using Newton's third law to require the forces and couples applied by the Ith element on the (I–1)th element to be equal in magnitude and opposite in direction to those applied by the (I–1)th element on the Ith element

$${}_{I-1}\mathbf{m}_2^i + {}_I\mathbf{m}_1^i = 0 \quad \text{for } I = 2, 3, \ldots, N \text{ and } i = 0, 1, 2. \tag{139}$$

Moreover, the boundary conditions are determined by specifying

$$\begin{aligned} &\{{}_1\mathbf{d}_0^* \text{ or } {}_1\mathbf{m}_1^0\} \quad \text{and} \quad \{{}_1\mathbf{d}_1^* \text{ or } {}_1\mathbf{m}_1^1\} \quad \text{and} \quad \{{}_1\mathbf{d}_2^* \text{ or } {}_1\mathbf{m}_1^2\}, \\ &\{{}_{N+1}\mathbf{d}_0^* \text{ or } {}_{N+1}\mathbf{m}_2^0\} \quad \text{and} \quad \{{}_{N+1}\mathbf{d}_1^* \text{ or } {}_{N+1}\mathbf{m}_2^1\} \quad \text{and} \quad \{{}_{N+1}\mathbf{d}_2^* \text{ or } {}_{N+1}\mathbf{m}_2^2\}. \end{aligned} \tag{140}$$

Again, mixed conditions or mixed–mixed conditions can be specified without difficulties.

By way of example, it is of interest to consider the case of a beam with rectangular cross-section for which the director inertia coefficients are given by Eqs. (122), (124) and (134), with subscripts I attached to each term, and the remaining values of ${}_Iy^{ij}$ vanishing. Moreover, using expression (70) and kinematic coupling conditions (138), it follows that

$$\begin{aligned} &{}_I\mathbf{w}_0 = \frac{1}{2}\left[{}_I\dot{\mathbf{d}}_0^* + {}_{I+1}\dot{\mathbf{d}}_0^*\right], \quad {}_I\mathbf{w}_1 = \frac{1}{2}\left[{}_I\dot{\mathbf{d}}_1^* + {}_{I+1}\dot{\mathbf{d}}_1^*\right], \quad {}_I\mathbf{w}_2 = \frac{1}{2}\left[{}_I\dot{\mathbf{d}}_2^* + {}_{I+1}\dot{\mathbf{d}}_2^*\right], \\ &{}_I\mathbf{w}_3 = \frac{1}{{}_IL}\left[{}_{I+1}\dot{\mathbf{d}}_0^* - {}_I\dot{\mathbf{d}}_0^*\right], \quad {}_I\mathbf{w}_4 = \frac{1}{{}_IL}\left[{}_{I+1}\dot{\mathbf{d}}_1^* - {}_I\dot{\mathbf{d}}_1^*\right], \quad {}_I\mathbf{w}_5 = \frac{1}{{}_IL}\left[{}_{I+1}\dot{\mathbf{d}}_2^* - {}_I\dot{\mathbf{d}}_2^*\right]. \end{aligned} \tag{141}$$

Then, Eq. (71) become

$$
\begin{aligned}
{}_I\mathbf{m}_1^0 &= {}_Im\left[\frac{1}{4}+\frac{{}_Iy^{33}}{{}_IL^2}\right]{}_I\ddot{\mathbf{d}}_0^* + {}_Im\left[\frac{1}{4}-\frac{{}_Iy^{33}}{{}_IL^2}\right]{}_{I+1}\ddot{\mathbf{d}}_0^* - \frac{1}{2}[{}_Im_I\mathbf{B}^0] + \frac{1}{{}_IL}[{}_Im_I\mathbf{B}^3] - \frac{1}{{}_IL}[{}_I\mathbf{t}^3],\\
{}_I\mathbf{m}_2^0 &= {}_Im\left[\frac{1}{4}-\frac{{}_Iy^{33}}{{}_IL^2}\right]{}_I\ddot{\mathbf{d}}_0^* + {}_Im\left[\frac{1}{4}+\frac{{}_Iy^{33}}{{}_IL^2}\right]{}_{I+1}\ddot{\mathbf{d}}_0^* - \frac{1}{2}[{}_Im_I\mathbf{B}^0] - \frac{1}{{}_IL}[{}_Im_I\mathbf{B}^3] + \frac{1}{{}_IL}[{}_I\mathbf{t}^3],\\
{}_I\mathbf{m}_1^1 &= {}_Im\left[\frac{{}_Iy^{11}}{4}+\frac{{}_Iy^{44}}{{}_IL^2}\right]{}_I\ddot{\mathbf{d}}_1^* + {}_Im\left[\frac{{}_Iy^{11}}{4}-\frac{{}_Iy^{44}}{{}_IL^2}\right]{}_{I+1}\ddot{\mathbf{d}}_1^* - \frac{1}{2}[{}_Im_I\mathbf{B}^1] + \frac{1}{{}_IL}[{}_Im_I\mathbf{B}^4] + \frac{1}{2}[{}_I\mathbf{t}^1] - \frac{1}{{}_IL}[{}_I\mathbf{t}^4],\\
{}_I\mathbf{m}_2^1 &= {}_Im\left[\frac{{}_Iy^{11}}{4}-\frac{{}_Iy^{44}}{{}_IL^2}\right]{}_I\ddot{\mathbf{d}}_1^* + {}_Im\left[\frac{{}_Iy^{11}}{4}+\frac{{}_Iy^{44}}{{}_IL^2}\right]{}_{I+1}\ddot{\mathbf{d}}_1^* - \frac{1}{2}[{}_Im_I\mathbf{B}^1] - \frac{1}{{}_IL}[{}_Im_I\mathbf{B}^4] + \frac{1}{2}[{}_I\mathbf{t}^1] + \frac{1}{{}_IL}[{}_I\mathbf{t}^4],\\
{}_I\mathbf{m}_1^2 &= {}_Im\left[\frac{{}_Iy^{22}}{4}+\frac{{}_Iy^{55}}{{}_IL^2}\right]{}_I\ddot{\mathbf{d}}_2^* + {}_Im\left[\frac{{}_Iy^{22}}{4}-\frac{{}_Iy^{55}}{{}_IL^2}\right]{}_{I+1}\ddot{\mathbf{d}}_2^* - \frac{1}{2}[{}_Im_I\mathbf{B}^2] + \frac{1}{{}_IL}[{}_Im_I\mathbf{B}^5] + \frac{1}{2}[{}_I\mathbf{t}^2] - \frac{1}{{}_IL}[{}_I\mathbf{t}^5],\\
{}_I\mathbf{m}_2^2 &= {}_Im\left[\frac{{}_Iy^{22}}{4}-\frac{{}_Iy^{55}}{{}_IL^2}\right]{}_I\ddot{\mathbf{d}}_2^* + {}_Im\left[\frac{{}_Iy^{22}}{4}+\frac{{}_Iy^{55}}{{}_IL^2}\right]{}_{I+1}\ddot{\mathbf{d}}_2^* - \frac{1}{2}[{}_Im_I\mathbf{B}^2] - \frac{1}{{}_IL}[{}_Im_I\mathbf{B}^5] + \frac{1}{2}[{}_I\mathbf{t}^2] + \frac{1}{{}_IL}[{}_I\mathbf{t}^5].
\end{aligned} \tag{142}
$$

To analyze the results of a particular problem, it is convenient to define the forces and couples ${}_I\mathbf{m}^{0*}$, ${}_I\mathbf{m}^{1*}$, ${}_I\mathbf{m}^{2*}$ applied to the Ith element (on its cross-section, which has an outward normal vector that makes an acute angle with the vector ${}_I\mathbf{d}_3$) using the expressions

$$
\begin{aligned}
&{}_1\mathbf{m}^{0*} = -{}_1\mathbf{m}_1^0, \qquad {}_1\mathbf{m}^{1*} = -{}_1\mathbf{m}_1^1, \qquad {}_1\mathbf{m}^{2*} = -{}_1\mathbf{m}_1^2,\\
&{}_1\mathbf{m}^{0*} = {}_{I-1}\mathbf{m}_2^0, \qquad {}_1\mathbf{m}^{1*} = {}_{I-1}\mathbf{m}_2^1, \qquad {}_1\mathbf{m}^{2*} = {}_{I-1}\mathbf{m}_2^2 \qquad \text{for } I = 2,3,\ldots,N+1.
\end{aligned} \tag{143}
$$

In these definitions, minus signs have been used for (${}_I\mathbf{m}^{0*}$, ${}_I\mathbf{m}^{1*}$, ${}_I\mathbf{m}^{2*}$) to take into account that the end ${}_1\partial P^*$ has an outward normal, which makes an obtuse angle with ${}_1\mathbf{d}_3$. Furthermore, it is convenient to use Eqs. (75) and (143) to define the moments ${}_I\mathbf{m}^*$ by the expressions

$$
{}_I\mathbf{m}^* = {}_I\mathbf{d}_1^* \times {}_I\mathbf{m}^{1*} + {}_I\mathbf{d}_2^* \times {}_I\mathbf{m}^{2*} \quad \text{for } I = 1,2,\ldots,N+1. \tag{144}
$$

Now, substitution of Eq. (142) into kinetic coupling equation (139) yields $3(N-1)$ ordinary vector differential equations for the $3(N+1)$ unknowns ${}_I\mathbf{d}_0^*$, ${}_I\mathbf{d}_1^*$, ${}_I\mathbf{d}_2^*$. Six additional vector equations are obtained by boundary condition (140). Moreover, since these differential equations are second order in time, they require initial conditions for the quantities

$$
\{{}_I\mathbf{d}_0^*, {}_I\mathbf{d}_1^*, {}_I\mathbf{d}_2^*\}, \quad \{{}_I\dot{\mathbf{d}}_0^*, {}_I\dot{\mathbf{d}}_1^*, {}_I\dot{\mathbf{d}}_2^*\} \qquad \text{at } t = 0. \tag{145}
$$

For the special case when the beam is uniform in its reference configuration and it is divided into elements of equal length L, then the subscript I can be omitted from the quantities L, m and y^{ij} so that

$$
{}_IL = L, \qquad {}_Im = m, \qquad {}_Iy^{ij} = y^{ij}. \tag{146}
$$

Moreover, for this case, the $3(N-1)$ vector kinetic coupling equation (139) can be written in the forms

$$
\begin{aligned}
&m\left[\left\{\frac{1}{4}-\frac{y^{33}}{L^2}\right\}{}_{I-1}\ddot{\mathbf{d}}_0^* + 2\left\{\frac{1}{4}+\frac{y^{33}}{L^2}\right\}{}_{I}\ddot{\mathbf{d}}_0^* + \left\{\frac{1}{4}-\frac{y^{33}}{L^2}\right\}{}_{I+1}\ddot{\mathbf{d}}_0^*\right] + m\left[-\frac{1}{2}\{{}_{I-1}\mathbf{B}^0 + {}_I\mathbf{B}^0\} - \frac{1}{L}\{{}_{I-1}\mathbf{B}^3 - {}_I\mathbf{B}^3\}\right] \\
&\quad + \frac{1}{L}\left[{}_{I-1}\mathbf{t}^3 - {}_I\mathbf{t}^3\right] = 0, \\
&m\left[\left\{\frac{y^{11}}{4}-\frac{y^{44}}{L^2}\right\}{}_{I-1}\ddot{\mathbf{d}}_1^* + 2\left\{\frac{y^{11}}{4}+\frac{y^{44}}{L^2}\right\}{}_{I}\ddot{\mathbf{d}}_1^* + \left\{\frac{y^{11}}{4}-\frac{y^{44}}{L^2}\right\}{}_{I+1}\ddot{\mathbf{d}}_1^*\right] + m\left[-\frac{1}{2}\{{}_{I-1}\mathbf{B}^1 + {}_I\mathbf{B}^1\} - \frac{1}{L}\{{}_{I-1}\mathbf{B}^4 - {}_I\mathbf{B}^4\}\right] \\
&\quad + \left[\frac{1}{2}\{{}_{I-1}\mathbf{t}^1 + {}_I\mathbf{t}^1\} + \frac{1}{L}\{{}_{I-1}\mathbf{t}^4 - {}_I\mathbf{t}^4\}\right] = 0, \\
&m\left[\left\{\frac{y^{22}}{4}-\frac{y^{55}}{L^2}\right\}{}_{I-1}\ddot{\mathbf{d}}_2^* + 2\left\{\frac{y^{22}}{4}+\frac{y^{55}}{L^2}\right\}{}_{I}\ddot{\mathbf{d}}_2^* + \left\{\frac{y^{22}}{4}-\frac{y^{55}}{L^2}\right\}{}_{I+1}\dot{\mathbf{d}}_2^*\right] + m\left[-\frac{1}{2}\{{}_{I-1}\mathbf{B}^2 + {}_I\mathbf{B}^2\} - \frac{1}{L}\{{}_{I-1}\mathbf{B}^5 - {}_I\mathbf{B}^5\}\right] \\
&\quad + \left[\frac{1}{2}\{I-1\mathbf{t}^2 + {}_I\mathbf{t}^2\} + \frac{1}{L}\{{}_{I-1}\mathbf{t}^5 - {}_I\mathbf{t}^5\}\right] = 0 \quad \text{for } I = 2,3,\ldots,N.
\end{aligned}
\tag{147}
$$

Then, with the help of Eqs. (142) and (143), the forces and moments become

$$
\begin{aligned}
{}_1\mathbf{m}^{0*} &= -m\left[\frac{1}{4}+\frac{y^{33}}{L^2}\right]{}_1\ddot{\mathbf{d}}_0^* - m\left[\frac{1}{4}-\frac{y^{33}}{L^2}\right]{}_2\ddot{\mathbf{d}}_0^* + m\left[\frac{1}{2}\{{}_1\mathbf{B}^0\} - \frac{1}{L}\{{}_1\mathbf{B}^3\}\right] + \frac{1}{L}\left[{}_1\mathbf{t}^3\right], \\
{}_1\mathbf{m}^{1*} &= -m\left[\frac{y^{11}}{4}+\frac{y^{44}}{L^2}\right]{}_1\ddot{\mathbf{d}}_1^* - m\left[\frac{y^{11}}{4}-\frac{y^{44}}{L^2}\right]{}_2\ddot{\mathbf{d}}_1^* + m\left[\frac{1}{2}\{{}_1\mathbf{B}^1\} - \frac{1}{L}\{{}_1\mathbf{B}^4\}\right] - \left[\frac{1}{2}\{{}_1\mathbf{t}^1\} - \frac{1}{L}\{{}_1\mathbf{t}^4\}\right], \\
{}_1\mathbf{m}^{2*} &= -m\left[\frac{y^{22}}{4}+\frac{y^{55}}{L^2}\right]{}_1\ddot{\mathbf{d}}_2^* - m\left[\frac{y^{22}}{4}-\frac{y^{55}}{L^2}\right]{}_2\ddot{\mathbf{d}}_2^* + m\left[\frac{1}{2}\{{}_1\mathbf{B}^2\} - \frac{1}{L}\{{}_1\mathbf{B}^5\}\right] - \left[\frac{1}{2}\{{}_1\mathbf{t}^2\} - \frac{1}{L}\{{}_1\mathbf{t}^5\}\right], \\
{}_I\mathbf{m}^{0*} &= m\left[\frac{1}{4}-\frac{y^{33}}{L^2}\right]{}_{I-1}\ddot{\mathbf{d}}_0^* + m\left[\frac{1}{4}+\frac{y^{33}}{L^2}\right]{}_I\ddot{\mathbf{d}}_0^* - m\left[\frac{1}{2}\{{}_{I-1}\mathbf{B}^0\} + \frac{1}{L}\{{}_{I-1}\mathbf{B}^3\}\right] + \frac{1}{L}\left[{}_{I-1}\mathbf{t}^3\right], \\
{}_I\mathbf{m}^{1*} &= m\left[\frac{y^{11}}{4}-\frac{y^{44}}{L^2}\right]{}_{I-1}\ddot{\mathbf{d}}_1^* + m\left[\frac{y^{11}}{4}+\frac{y^{44}}{L^2}\right]{}_I\ddot{\mathbf{d}}_1^* - m\left[\frac{1}{2}\{{}_{I-1}\mathbf{B}^1\} + \frac{1}{L}\{{}_{I-1}\mathbf{B}^4\}\right] + \left[\frac{1}{2}\{{}_{I-1}\mathbf{t}^1\} + \frac{1}{L}\{{}_{I-1}\mathbf{t}^4\}\right], \\
{}_I\mathbf{m}^{2*} &= m\left[\frac{y^{22}}{4}-\frac{y^{55}}{L^2}\right]{}_{I-1}\ddot{\mathbf{d}}_2^* + m\left[\frac{y^{22}}{4}+\frac{y^{55}}{L^2}\right]{}_I\ddot{\mathbf{d}}_2^* - m\left[\frac{1}{2}\{{}_{I-1}\mathbf{B}^2\} + \frac{1}{L}\{{}_{I-1}\mathbf{B}^5\}\right] + \left[\frac{1}{2}\{{}_{I-1}\mathbf{t}^2\} + \frac{1}{L}\{{}_{I-1}\mathbf{t}^5\}\right] \\
&\quad \text{for } I = 2,3,\ldots,N+1.
\end{aligned}
\tag{148}
$$

Also, the six vector equations characterizing boundary condition (140) are determined by specifying

$$
\begin{aligned}
&\{{}_1\mathbf{d}_0^* \text{ or } {}_1\mathbf{m}^{0*}\} \quad \text{and} \quad \{{}_1\mathbf{d}_1^* \text{ or } {}_1\mathbf{m}^{1*}\} \quad \text{and} \quad \{{}_1\mathbf{d}_2^* \text{ or } {}_1\mathbf{m}^{2*}\}, \\
&\{{}_{N+1}\mathbf{d}_0^* \text{ or } {}_{N+1}\mathbf{m}^{0*}\} \quad \text{and} \quad \{{}_{N+1}\mathbf{d}_1^* \text{ or } {}_{N+1}\mathbf{m}^{1*}\} \quad \text{and} \quad \{{}_{N+1}\mathbf{d}_2^* \text{ or } {}_{N+1}\mathbf{m}^{2*}\}.
\end{aligned}
\tag{149}
$$

In particular, it is noted that Eqs. (147) and (148) have the usual tridiagonal symmetric form for the mass matrix used to determine the accelerations of the directors $\{{}_I\mathbf{d}_0^*, {}_I\mathbf{d}_1^*, {}_I\mathbf{d}_2^*\}$. Also, it can be seen from Eq. (147) that, if y^{33}, y^{44} and y^{55} are specified by

$$
y^{33} = \frac{L^2}{4}, \qquad y^{44} = \frac{L^2 y^{11}}{4}, \qquad y^{55} = \frac{L^2 y^{22}}{4},
\tag{150}
$$

instead of the values in Eqs. (124) and (134), then Eq. (147) are diagonalized in terms of the accelerations. However, such a specification causes an increased number of elements to be used for the same accuracy. Specifications of the director inertia coefficient of this type were considered previously (Rubin and Gottlieb, 1996) when modeling the dynamics of strings.

10. A nonlinear example problem

As an example, consider the problem of a uniform cantilever beam (of rectangular cross-section), which is deformed in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane by a force applied to one of its ends (Fig. 6). The length of the beam in its reference configuration is taken to be L_0 and the beam is divided into N equal segments so that the length L of each segment is given by

$$L = \frac{L_0}{N}. \tag{151}$$

In the absence of accelerations, body forces and tractions on the beam's lateral surface (${}_I\mathbf{B}^i = 0$), kinetic coupling equation (147) require

$$\begin{aligned}
&\frac{1}{L}[{}_{I-1}\mathbf{t}^3 - {}_I\mathbf{t}^3] = 0,\\
&\left[\frac{1}{2}\{{}_{I-1}\mathbf{t}^1 + {}_I\mathbf{t}^1\} + \frac{1}{L}\{{}_{I-1}\mathbf{t}^4 - {}_I\mathbf{t}^4\}\right] = 0,\\
&\left[\frac{1}{2}\{{}_{I-1}\mathbf{t}^2 + {}_I\mathbf{t}^2\} + \frac{1}{L}\{{}_{I-1}\mathbf{t}^5 - {}_I\mathbf{t}^5\}\right] = 0 \quad \text{for } I = 2, 3, \ldots, N.
\end{aligned} \tag{152}$$

Also, expression (148) for the forces and moments reduce to

$$\begin{aligned}
&{}_1\mathbf{m}^{0*} = \frac{1}{L}[{}_1\mathbf{t}^3], \qquad {}_1\mathbf{m}^{1*} = -\left[\frac{1}{2}\{{}_1\mathbf{t}^1\} - \frac{1}{L}\{{}_1\mathbf{t}^4\}\right],\\
&{}_1\mathbf{m}^{2*} = -\left[\frac{1}{2}\{{}_1\mathbf{t}^2\} - \frac{1}{L}\{{}_1\mathbf{t}^5\}\right],\\
&{}_I\mathbf{m}^{0*} = \frac{1}{L}[{}_{I-1}\mathbf{t}^3], \qquad {}_I\mathbf{m}^{1*} = \left[\frac{1}{2}\{{}_{I-1}\mathbf{t}^1\} + \frac{1}{L}\{{}_{I-1}\mathbf{t}^4\}\right],\\
&{}_I\mathbf{m}^{2*} = \left[\frac{1}{2}\{{}_{I-1}\mathbf{t}^2\} + \frac{1}{L}\{{}_{I-1}\mathbf{t}^5\}\right] \quad \text{for } I = 2, 3, \ldots, N+1.
\end{aligned} \tag{153}$$

Since the centerline of the beam remains in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane, the kinematics are specified by the $5(N+1)$ variables $({}_Ix_1^*,\ {}_Ix_3^*,\ {}_I\theta,\ {}_I\lambda_1,\ {}_I\lambda_2)$

$$\begin{aligned}
&{}_I\mathbf{d}_0^* = {}_Ix_1^*\mathbf{e}_1 + {}_Ix_3^*\mathbf{e}_3, \qquad {}_I\mathbf{d}_1^* = {}_I\lambda_1[\cos({}_I\theta)\mathbf{e}_1 - \sin({}_I\theta)\mathbf{e}_3],\\
&{}_I\mathbf{d}_2^* = {}_I\lambda_2\mathbf{e}_2 \quad \text{for } I = 1, 2, \ldots, N+1.
\end{aligned} \tag{154}$$

Also, in the reference configuration, these variables are given by

$${}_Ix_1^* = 0, \qquad {}_Ix_3^* = \frac{(I-1)L_0}{N}, \qquad {}_I\lambda_1 = 1, \qquad {}_I\theta = 0, \qquad {}_I\lambda_2 = 1, \quad \text{for } I = 1, 2, \ldots, N+1. \tag{155}$$

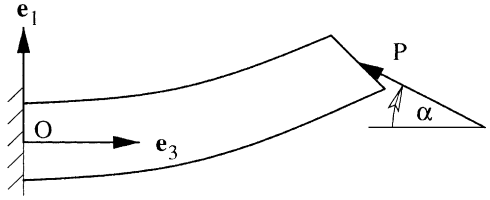

Fig. 6. Sketch of a cantilever beam "clamped" at one end and loaded by a force P acting at an angle α at the other end.

Now, the boundary conditions are specified by

$$
\begin{aligned}
&{}_1x_1^* = 0, \quad {}_1x_3^* = 0, \quad {}_1\mathbf{m}^{1*}\cdot\mathbf{e}_1 = 0, \quad {}_1\theta = 0, \quad {}_1\mathbf{m}^{2*}\cdot\mathbf{e}_2 = 0,\\
&{}_{N+1}\mathbf{m}^{0*}\cdot\mathbf{e}_1 = P\sin(\alpha), \qquad {}_{N+1}\mathbf{m}^{0*}\cdot\mathbf{e}_3 = -P\cos(\alpha),\\
&{}_{N+1}\mathbf{m}^{1*}\cdot\mathbf{e}_1 = 0, \qquad {}_{N+1}\mathbf{m}^{1*}\cdot\mathbf{e}_3 = 0, \qquad {}_{N+1}\mathbf{m}^{2*}\cdot\mathbf{e}_2 = 0,
\end{aligned}
\tag{156}
$$

where the magnitude P and angle α characterize the force applied to the end $I = N+1$. In particular, notice that the cross-section at $I = 1$ is not allowed to rotate and its center is held fixed but the cross-section is allowed to change its dimensions (since ${}_1\lambda_1$ and ${}_1\lambda_2$ need not remain unity) so that it is only partially "clamped".

For this problem, Eq. (68) with subscript I yields

$$
{}_I\mathbf{D}_1 = \mathbf{e}_1, \qquad {}_I\mathbf{D}_2 = \mathbf{e}_2, \qquad {}_I\mathbf{D}_3 = \mathbf{e}_3, \qquad {}_I\mathbf{D}_4 = 0, \qquad {}_I\mathbf{D}_5 = 0,
\tag{157}
$$

and Eq. (69) with subscript I yields expressions for ${}_I\mathbf{d}_i$

$$
\begin{aligned}
&{}_I\mathbf{d}_0 = \frac{1}{2}[{}_I\mathbf{d}_0^* + {}_{I+1}\mathbf{d}_0^*], \qquad {}_I\mathbf{d}_3 = \frac{1}{L}[{}_{I+1}\mathbf{d}_0^* - {}_I\mathbf{d}_0^*],\\
&{}_I\mathbf{d}_1 = \frac{1}{2}[{}_I\mathbf{d}_1^* + {}_{I+1}\mathbf{d}_1^*], \qquad {}_I\mathbf{d}_4 = \frac{1}{L}[{}_{I+1}\mathbf{d}_1^* - {}_I\mathbf{d}_1^*],\\
&{}_I\mathbf{d}_2 = \frac{1}{2}[{}_I\mathbf{d}_2^* + {}_{I+1}\mathbf{d}_2^*], \qquad {}_I\mathbf{d}_5 = \frac{1}{L}[{}_{I+1}\mathbf{d}_2^* - {}_I\mathbf{d}_2^*].
\end{aligned}
\tag{158}
$$

Also, with the help of Eqs. (16), (18), (24), (37), (61), (65) and (80), the relevant kinematic variables are given by

$$
\begin{aligned}
&{}_I\mathbf{F} = {}_I\mathbf{d}_1\otimes\mathbf{e}_1 + {}_I\mathbf{d}_2\otimes\mathbf{e}_2 + {}_I\mathbf{d}_3\otimes\mathbf{e}_3, \qquad {}_IJ = {}_I\mathbf{d}_1\times{}_I\mathbf{d}_2\cdot{}_I\mathbf{d}_3,\\
&{}_I\mathbf{B} = {}_I\mathbf{F}{}_I\mathbf{F}^{\mathrm{T}}, \qquad {}_I\boldsymbol{\beta}_1 = {}_I\mathbf{F}^{-1}{}_I\mathbf{d}_4, \qquad {}_I\boldsymbol{\beta}_2 = {}_I\mathbf{F}^{-1}{}_I\mathbf{d}_5,\\
&{}_I\kappa_1^3 = \mathbf{e}_3\cdot{}_I\boldsymbol{\beta}_1, \qquad {}_I\kappa_1^1 = \mathbf{e}_1\cdot{}_I\boldsymbol{\beta}_1, \qquad {}_I\kappa_2^2 = \mathbf{e}_2\cdot{}_I\boldsymbol{\beta}_2,\\
&{}_I\omega_1 = 0, \qquad {}_I\omega_2 = 0,
\end{aligned}
\tag{159}
$$

and the constitutive equations become

$$
\begin{aligned}
&{}_I\mathbf{T} = VK^*[{}_IJ - 1]\mathbf{I} + {}_IJ^{-2/3}V\mu^*\left[{}_I\mathbf{B} - \frac{1}{3}({}_I\mathbf{B}\cdot\mathbf{I})\mathbf{I}\right],\\
&{}_I\mathbf{t}^4 = VL[k_4({}_I\kappa_1^1){}_I\mathbf{d}^1 + k_1({}_I\kappa_1^3){}_I\mathbf{d}^3], \qquad {}_I\mathbf{t}^5 = VL[k_5({}_I\kappa_2^2){}_I\mathbf{d}^2],\\
&{}_I\mathbf{t}^i = [{}_I\mathbf{T} - {}_I\mathbf{t}^4\otimes{}_I\mathbf{d}_4 - {}_I\mathbf{t}^5\otimes{}_I\mathbf{d}_5]\cdot{}_I\mathbf{d}^i \quad \text{for } i = 1,2,3,
\end{aligned}
\tag{160}
$$

where ${}_I\mathbf{d}^i (i = 1,2,3)$ are the reciprocal vectors of ${}_I\mathbf{d}_i$.

Moreover, for the example problems to be considered, the values of μ^* and v^* are specified by Eq. (89), and the values of k_1, k_4 and k_5 are given by Eqs. (100) and (121), with k being specified

$$
k = 0.001.
\tag{161}
$$

Also, to emphasize the influence of shear deformation, the beam is taken to be reasonably thick with

$$
L_0 = 1\ \text{m}, \qquad H = W = 0.1\ \text{m}, \qquad A = HW = 0.01\ \text{m}^2.
\tag{162}
$$

The computer code MATLAB was used to program the equations in tensorial form and the subroutine fsolve was used to solve the $5(N+1)$ scalar equations characterizing equilibrium of planar deformation of a rod with N elements. Moreover, the initial guess for the equilibrium configuration was determined by solving the equations of an elastica, which are reviewed in Appendix E.

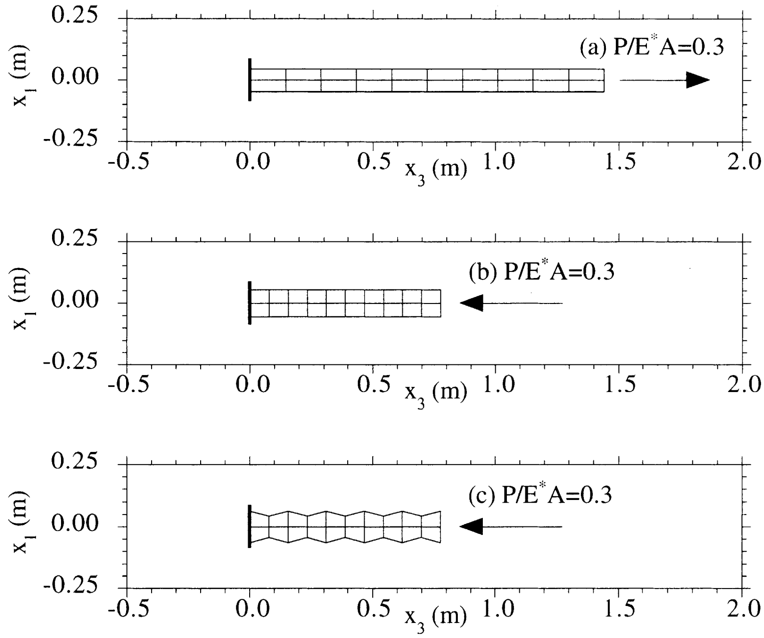

Fig. 7. Cosserat solutions for uniaxial stress for $P/E^*A = 0.3$. The correct solutions for (a) tension and (b) compression are obtained using $k = 0.001$ and the solution for (c) compression with hour glassing is obtained using $k = 0$. The number of elements $N = 10$ is taken for ease of interpretation because the unstressed elements are square.

Fig. 7 shows the solution for uniaxial stress with a nominal load P, which for the linear theory would cause 30% extension (Fig. 7(a) with $\alpha = 180°$) or compression (Fig. 7(b), (c) with $\alpha = 0°$). Since the solution is associated with uniform deformation, a single Cosserat point will reproduce the exact solution. However, 10 elements have been used to emphasize the change of axial and lateral dimensions of the Cosserat theory relative to the unstressed reference configuration, where the elements would be square in these plots. Fig. 7(a) shows that the rod extends more than 30% since the cross-section contracts, and Fig. 7(b) shows that the rod contracts less than 30% since the cross-section expands. Fig. 7(c) shows that when the value of k in Eq. (161) vanishes, then the solution admits an uncontrolled hour glassing mode. Therefore, for the remainder of the solutions presented, the value of k is specified by Eq. (161).

Fig. 8 shows the solution for a vertical force ($\alpha = 90°$) for different values of the nominal engineering shear strain P/μ^*A. Actually, the load corresponds to shearing only for small deformations, since for large deformations ($P/\mu^*A = 0.05$), the load rapidly transitions from shearing to tension as the beam is bent. The Cosserat solution compares quite well with the elastica solution, except for the highest load where the effects of shear deformation and extension in the Cosserat solution are apparent. Fig. 9 shows three equilibrium solutions for each of five different vertical loads ($\alpha = 90°$). The solution associated with branch I would be obtained by continuously increasing the vertical load from zero. Whereas the solutions for branches II and III require the temporary application of a bending moment until the equilibrium position is assumed with only a vertical force applied. Notice that at the nominal load of about 0.022 (Fig. 9(a)), branches II and III nearly coincide. Also, notice that as the load is increased, branches II and III diverge with a rapid change occurring near the minimal load 0.022. Comparison with the elastica solution shows that the effects of shear deformation in the Cosserat solution are quite significant, with the Cosserat beam being more flexible than the elastica. In particular, the Cosserat solution associated with branch III, for the highest load (Fig. 9(e)), indicates that the beam makes contact with itself (which was not included in the

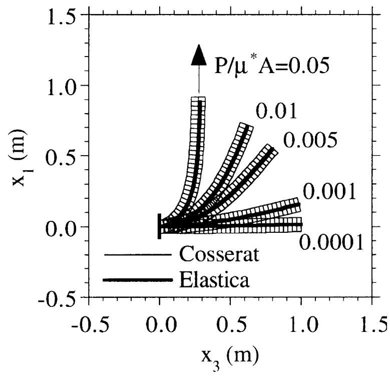

Fig. 8. Equilibrium solutions for a vertical force P applied to the end of a cantilever beam. Comparison of the elastica model with the Cosserat model for $N = 32$ elements.

solution procedure). The character of these three branches are shown more clearly in Fig. 10 where the load is plotted as a function of the curvature $\kappa(0)$ at the clamped end of the beam. For the Cosserat solution, this value of curvature was specified by

$$\kappa(0) = \frac{{}_1\mathbf{m}^* \cdot \mathbf{e}_2}{E^* I^*}, \tag{163}$$

where I^* is the second moment of area of the cross section given by fifth part of Eq. (E.3). In particular, notice that for a nominal load above about 0.022, equilibrium solutions associated with branches I, II and III are possible. However, the stability of these solutions has not been investigated.

The solutions given in Figs. 8–10 have been shown for 32 elements. Fig. 11 shows that reasonable convergence is obtained for 32 elements. In particular, for the shearing branch I, Fig. 11(a) shows that four elements are sufficient for a nominal load of 0.001, and Fig. 11(b) shows that eight elements are sufficient for a nominal load of 0.01. Also, Fig. 11(c) shows that 32 elements are required to obtain good convergence for the highest load associated with branch II.

Finally, compressive ($\alpha = 0°$) buckling of the beam is considered in Fig. 12, where the horizontal load P has been normalized by the linear Bernoulli–Euler critical load P_{cr}

$$P_{cr} = \frac{\pi^2 E^* I^*}{4 L_0^2}. \tag{164}$$

This figure shows a rapid change in the equilibrium configuration near the buckling load. It also shows that the elastica solution is quite accurate, except near the highest two loads where the effects of shear deformation become important.

11. Summary

The theory of a Cosserat point has been developed as a continuum theory with basic balance laws that characterize: conservation of mass, balance of linear momentum, balances of director momentum and balance of angular momentum. Constitutive equations for nonlinear elastic Cosserat points have been derived in a similar manner to those in the full three-dimensional theory by relating resultant forces and couples to derivatives of a strain energy function. Moreover, the resulting equations are properly invariant under SRBM and they are valid for large deformations and rotations. Also, the constitutive equations have

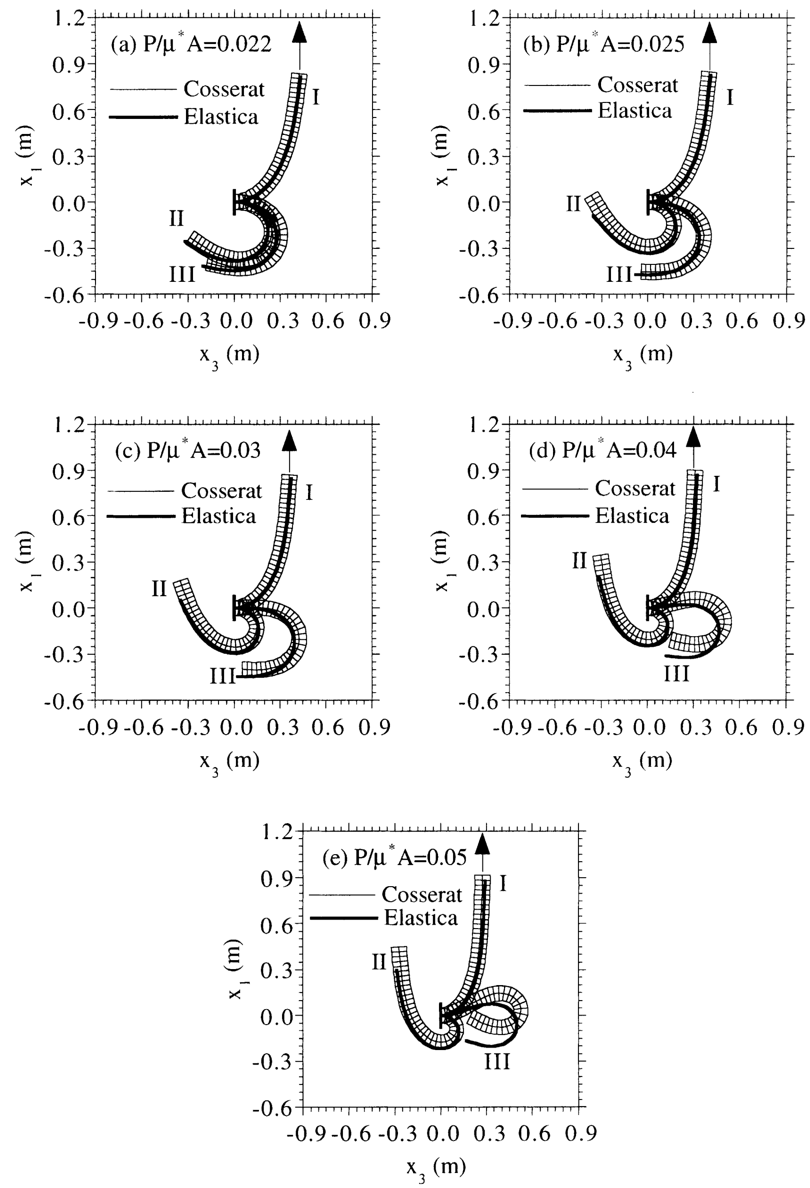

Fig. 9. Three equilibrium solutions for a vertical force P applied to the end of a cantilever beam. Comparison of the elastica model with the Cosserat model for $N = 32$ elements.

been suitably restricted so as to produce exact solutions for all homogeneous deformations of the rod. In addition, the constitutive equations for inhomogeneous deformations (like those associated with bending and torsion) have been considered within the context of a strain energy function, which is a quadratic function of nonlinear strain measures.

The numerical solution procedure proposed in this paper models dynamic nonlinear three-dimensional motion of a rod by considering N connected Cosserat points, each of which models the deformation of a section of the rod. For dynamic problems, the balance laws and boundary conditions reduce to a finite set

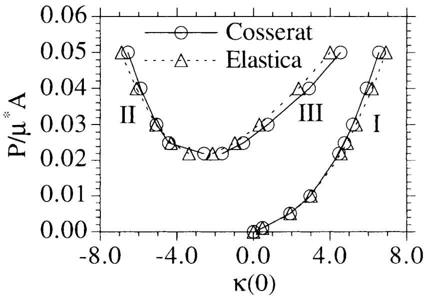

Fig. 10. Plots of the load versus the curvature $\kappa(0)$ at the "clamped" end of a cantilever beam loaded by a vertical force P. Comparison of the elastica model with the Cosserat model for $N = 32$ elements.

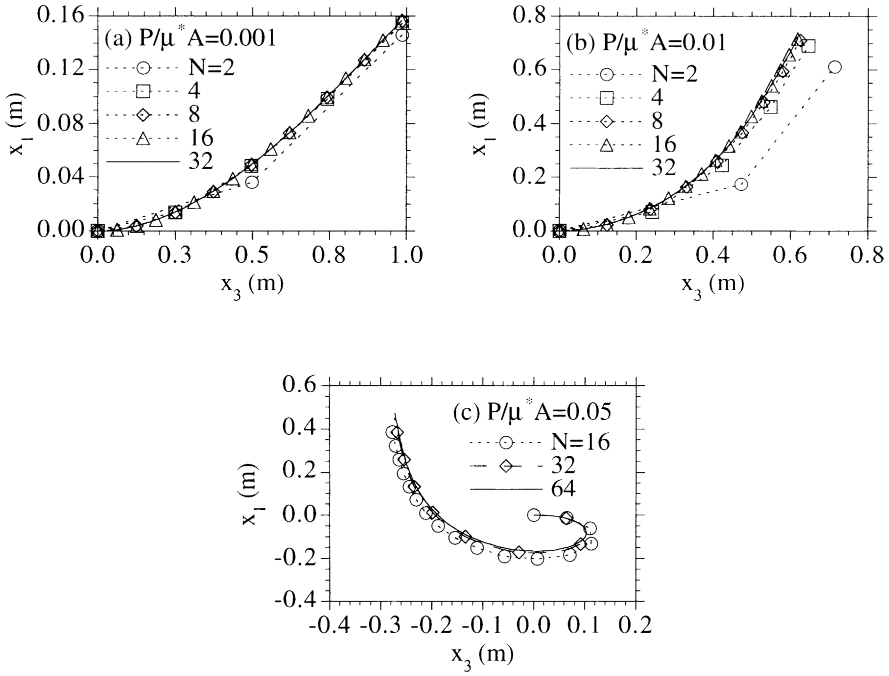

Fig. 11. Center line of a cantilever beam loaded by a vertical force P. Convergence of the solution for different numbers N of elements. (a) and (b) show the solutions of type I and (c) shows the solution of type II.

of ordinary differential equations that depend on time only, whereas for equilibrium problems the resulting equations are algebraic.

In contrast with the standard Galerkin procedure, the constitutive equations of the Cosserat theory take forms very similar to those in the three-dimensional theory, and the constitutive coefficients are determined by comparison with known exact solutions of the three-dimensional theory or with appropriate experiments. This allows for the determination of constitutive coefficients, which take full advantage of the reduced number of degrees of freedom of the model.

The solutions of a number of static problems have been considered. For these problems, it has been shown that the results of the Cosserat theory compare well with those of an elastica, except where the effects

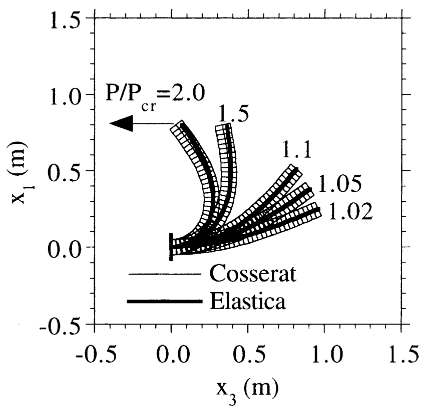

Fig. 12. Buckling due to a compressive horizontal force P. The values of P have been normalized by the critical Bernoulli–Euler buckling load P_{cr}.

of shear deformation or extension become important. In these later cases, the Cosserat theory predicts results that are consistent with the added flexibility associated with the additional modes of deformation included in the Cosserat theory.

Acknowledgements

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Appendix A. Derivation from the three-dimensional theory

A rod-like structure is a three-dimensional body that can be modeled as a space curve with some small cross-sectional area. In its reference configuration, material points on the space curve are parameterized by the convected coordinate θ^3, and the cross-sections ($\theta^3 =$ constant) are taken to be planes, which are parameterized by the convected coordinates θ^α, with $\theta^\alpha = 0$ being the centroid of the cross-section.

In its reference configuration, a section of the rod occupies the region of space P_0 that is bounded by its lateral surface ∂P_{0L} and its ends ∂P_{01} (with $\theta^3 = \xi_1$) and ∂P_{02} (with $\theta^3 = \xi_2$). Also, material points in the reference configuration of this section of the rod are located by the position vector $\mathbf{X}^*(\theta^i)$, such that

$$\mathbf{X}^*(\theta^i) = \mathbf{D}_0 + \theta^1[\mathbf{D}_1 + \bar{\theta}^3\mathbf{D}_4] + \theta^2[\mathbf{D}_2 + \bar{\theta}^3\mathbf{D}_5] + \bar{\theta}^3\mathbf{D}_3,$$

$$\bar{\theta}^3 = \theta^3 - \frac{1}{2}(\xi_1 + \xi_2),$$

$$\begin{aligned} &\mathbf{G}_1 = \mathbf{X}^*_{,1} = \mathbf{D}_1 + \bar{\theta}^3\mathbf{D}_4, \qquad \mathbf{G}_2 = \mathbf{X}^*_{,2} = \mathbf{D}_2 + \bar{\theta}^3\mathbf{D}_5, \\ &\mathbf{G}_3 = \mathbf{X}^*_{,3} = \mathbf{D}_3 + \theta^1\mathbf{D}_4 + \theta^2\mathbf{D}_5, \qquad G^{1/2} = \mathbf{G}_1 \times \mathbf{G}_2 \cdot \mathbf{G}_3 > 0, \end{aligned} \tag{A.1}$$

where the vectors $\mathbf{D}_i$ are constant vectors, the vectors ($\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3$) are assumed to be linearly independent (14), and $\mathbf{G}_i$ are the covariant base vectors in the reference configuration.

In the present configuration, the region occupied by this material section of the rod is denoted by P, with the lateral surface ∂P_L and the ends ∂P_1 and ∂P_2. Material points in the present configuration of this section of the rod are located by the position vector $\mathbf{x}^*(\theta^i,t)$, which is a function of the convected coordinates and time t. In order to motivate the balance laws for the Cosserat point model of a rod, it is assumed that $\mathbf{x}^*$ admits a similar representation to Eq. (A.1), such that

$$\begin{aligned} &\mathbf{x}^*(\theta^i,t) = \mathbf{d}_0(t) + \theta^1[\mathbf{d}_1(t) + \bar{\theta}^3\mathbf{d}_4(t)] + \theta^2[\mathbf{d}_2(t) + \bar{\theta}^3\mathbf{d}_5(t)] + \bar{\theta}^3\mathbf{d}_3(t), \\ &\mathbf{g}_1 = \mathbf{x}^*_{,1} = \mathbf{d}_1 + \bar{\theta}^3\mathbf{d}_4, \qquad \mathbf{g}_2 = \mathbf{x}^*_{,2} = \mathbf{d}_2 + \bar{\theta}^3\mathbf{d}_5, \\ &\mathbf{g}_3 = \mathbf{x}^*_{,3} = \mathbf{d}_3 + \theta^1\mathbf{d}_4 + \theta^2\mathbf{d}_5, \qquad g^{1/2} = \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 > 0, \end{aligned} \tag{A.2}$$

where $\mathbf{d}_i(t)$ are vector functions of time only, the vectors $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ are assumed to be linearly independent (2), and $\mathbf{g}_i$ are the covariant base vectors in the present configuration.

Now, the local forms of the conservation of mass and the balances of linear and angular momentum can be written as

$$\begin{aligned} &m^* = \rho^* g^{1/2} = \rho_0^* G^{1/2} = m^*(\theta^i), \\ &m^*\dot{\mathbf{v}}^* = m^*\mathbf{b}^* + \mathbf{t}^{*i}_{,i}, \qquad \mathbf{T}^{*\mathrm{T}} = \mathbf{T}^*, \end{aligned} \tag{A.3}$$

where ρ^* is the mass density per unit volume $\mathrm{d}v^*$ in the present configuration, ρ_0^* is the mass density per unit volume $\mathrm{d}V^*$ in the reference configuration, $\mathbf{v}^*$ is the material velocity, $\mathbf{b}^*$ is the specific (per unit mass) body force, $\mathbf{T}^*$ is the Cauchy stress, and $\mathbf{t}^{*i}$ are defined by

$$\mathbf{t}^{*i} = g^{1/2}\mathbf{T}^*\mathbf{g}^i. \tag{A.4}$$

Also, $\mathbf{G}^i$ and $\mathbf{g}^i$ are reciprocal vectors defined by

$$\mathbf{G}_i \cdot \mathbf{G}^j = \delta_i^j, \qquad \mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j \quad \text{for } i,j = 1,2,3, \tag{A.5}$$

and $\mathbf{t}^*$ is the stress vector per unit present area $\mathrm{d}a^*$, which is related to $\mathbf{T}^*$ and the unit outward normal vector $\mathbf{n}^*$ to the boundary ∂P of the region P by the expression

$$\mathbf{t}^* = \mathbf{T}^* n^*. \tag{A.6}$$

Next, balance law (A.3) (second and third part) can be multiplied by a weighting function ϕ of θ^i only, and the results can be integrated over the region P to deduce the averaged balance laws

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_P \phi\rho^*\mathrm{d}v^* = 0, \\ &\frac{\mathrm{d}}{\mathrm{d}t}\int_P \phi\rho^*\mathbf{v}^*\mathrm{d}v^* = \int_P \phi\rho^*\mathbf{b}^*\mathrm{d}v^* + \int_{\partial P_\mathrm{L}} \phi\mathbf{t}^*\mathrm{d}a^* - \int_P \mathbf{T}^*\mathbf{g}^i\phi_{,i}\mathrm{d}v^* + \int_{\partial P_1} \phi\mathbf{t}^*\mathrm{d}a^* + \int_{\partial P_2} \phi\mathbf{t}^*\mathrm{d}a^*. \end{aligned} \tag{A.7}$$

Also, the global form of the balance of angular momentum can be recorded as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_P \mathbf{x}^* \times \rho^*\mathbf{v}^*\mathrm{d}v^* = \int_P \mathbf{x}^* \times \rho^*\mathbf{b}^*\mathrm{d}v^* + \int_{\partial P} \mathbf{x}^* \times \mathbf{t}^*\mathrm{d}a^*. \tag{A.8}$$

Then, using the definitions in Appendix B, the conservation of mass (5) (first part) and the balances of director momentum (5) (second part) can be determined by Eq. (A.7) (first part) with $\phi = 1$, and Eq. (A.7) (second part) with $\phi = (1, \theta^\alpha, \bar{\theta}^3, \theta^\alpha\bar{\theta}^3)$. Also, the balance of angular momentum (7) for the Cosserat point can be obtained using the definitions in Appendix B and balance law (A.8).

Appendix B. Definitions for a Cosserat point

In this appendix, a number of definitions are introduced that are used to derive the balance laws for a Cosserat point from the three-dimensional theory. Specifically, the total mass m of the Cosserat point and the director inertia coefficients y^{ij} are defined by

$$
\begin{aligned}
& m = \int_P \rho^* \mathrm{d}v^*, \qquad y^{00} = 1, \qquad m y^{0\alpha} = \int_P \theta^\alpha \rho^* \mathrm{d}v^*, \\
& m y^{03} = \int_P \bar{\theta}^3 \rho^* \mathrm{d}v^*, \qquad m y^{04} = \int_P \theta^1 \bar{\theta}^3 \rho^* \mathrm{d}v^*, \\
& m y^{05} = \int_P \theta^2 \bar{\theta}^3 \rho^* \mathrm{d}v^*, \qquad m y^{\alpha\beta} = \int_P \theta^\alpha \theta^\beta \rho^* \mathrm{d}v^*, \\
& m y^{\alpha 3} = \int_P \theta^\alpha \bar{\theta}^3 \rho^* \mathrm{d}v^*, \qquad m y^{\alpha 4} = \int_P \theta^\alpha \theta^1 \bar{\theta}^3 \rho^* \mathrm{d}v^*, \\
& m y^{\alpha 5} = \int_P \theta^\alpha \theta^2 \bar{\theta}^3 \rho^* \mathrm{d}v^*, \qquad m y^{33} = \int_P \bar{\theta}^3 \bar{\theta}^3 \rho^* \mathrm{d}v^*, \\
& m y^{34} = \int_P \theta^1 \bar{\theta}^3 \bar{\theta}^3 \rho^* \mathrm{d}v^*, \qquad m y^{35} = \int_P \theta^2 \bar{\theta}^3 \bar{\theta}^3 \rho^* \mathrm{d}v^*, \\
& m y^{44} = \int_P \theta^1 \theta^1 \bar{\theta}^3 \bar{\theta}^3 \rho^* \mathrm{d}v^*, \qquad m y^{45} = \int_P \theta^1 \theta^2 \bar{\theta}^3 \bar{\theta}^3 \rho^* \mathrm{d}v^*, \\
& m y^{55} = \int_P \theta^2 \theta^2 \bar{\theta}^3 \bar{\theta}^3 \rho^* \mathrm{d}v^*, \qquad y^{ij} = y^{ji} \quad \text{for } i, j = 0, 1, 2, \ldots, 5.
\end{aligned}
\tag{B.1}
$$

The assigned fields $\mathbf{B}^i$ due to body force and tractions on the lateral surface of the section of the rod are defined by

$$
\begin{aligned}
& m\mathbf{B}^0 = \int_P \rho^* \mathbf{b}^* \mathrm{d}v^* + \int_{\partial P_\mathrm{L}} \mathbf{t}^* \mathrm{d}a^*, \\
& m\mathbf{B}^\alpha = \int_P \theta^\alpha \rho^* \mathbf{b}^* \mathrm{d}v^* + \int_{\partial P_\mathrm{L}} \theta^\alpha \mathbf{t}^* \mathrm{d}a^*, \\
& m\mathbf{B}^3 = \int_P \bar{\theta}^3 \rho^* \mathbf{b}^* \mathrm{d}v^* + \int_{\partial P_\mathrm{L}} \bar{\theta}^3 \mathbf{t}^* \mathrm{d}a^*, \\
& m\mathbf{B}^4 = \int_P \theta^1 \bar{\theta}^3 \rho^* \mathbf{b}^* \mathrm{d}v^* + \int_{\partial P_\mathrm{L}} \theta^1 \bar{\theta}^3 \mathbf{t}^* \mathrm{d}a^*, \\
& m\mathbf{B}^5 = \int_P \theta^2 \bar{\theta}^3 \rho^* \mathbf{b}^* \mathrm{d}v^* + \int_{\partial P_\mathrm{L}} \theta^2 \bar{\theta}^3 \mathbf{t}^* \mathrm{d}a^*.
\end{aligned}
\tag{B.2}
$$

Next, the director couples $\mathbf{m}_\alpha^i$ applied to the ends ∂P_α of the section of the rod are defined by

$$
\begin{aligned}
& \mathbf{m}_\alpha^0 = \int_{\partial P_\alpha} \mathbf{t}^* \mathrm{d}a^*, \qquad \mathbf{m}_\alpha^\beta = \int_{\partial P_\alpha} \theta^\beta \mathbf{t}^* \mathrm{d}a^*, \\
& \mathbf{m}_\alpha^3 = \int_{\partial P_\alpha} \bar{\theta}^3 \mathbf{t}^* \mathrm{d}a^*, \qquad \mathbf{m}_\alpha^4 = \int_{\partial P_\alpha} \theta^1 \bar{\theta}^3 \mathbf{t}^* \mathrm{d}a^*, \\
& \mathbf{m}_\alpha^5 = \int_{\partial P_\alpha} \theta^2 \bar{\theta}^3 \mathbf{t}^* \mathrm{d}a^*,
\end{aligned}
\tag{B.3}
$$

and the external assigned fields $\mathbf{b}^i$ due to body force and surface tractions on the lateral surface and on the ends of the rod are defined by

$$
m\mathbf{b}^i = m\mathbf{B}^i + \mathbf{m}_1^i + \mathbf{m}_2^i \quad \text{for} \quad i = 0, 1, \ldots, 5. \tag{B.4}
$$

Furthermore, the intrinsic director couples $\mathbf{t}^i$ are defined by

$$\mathbf{t}^0 = 0, \qquad \mathbf{t}^\alpha = \int_P \mathbf{T}^* \mathbf{g}^\alpha \mathrm{d}v^*,$$
$$\mathbf{t}^3 = \int_P \mathbf{T}^* \mathbf{g}^3 \mathrm{d}v^*, \qquad \mathbf{t}^4 = \int_P \mathbf{T}^*[\mathbf{g}^1 \overline{\theta}^3 + \mathbf{g}^3 \theta^1] \mathrm{d}v^*, \qquad \mathbf{t}^5 = \int_P \mathbf{T}^*[\mathbf{g}^2 \overline{\theta}^3 + \mathbf{g}^3 \theta^2] \mathrm{d}v^*. \tag{B.5}$$

Now, substituting Eq. (B.5) in definition (9), it can be shown that

$$d^{1/2}\mathbf{T} = \int_P \mathbf{T}^*[\mathbf{g}^1 \otimes (\mathbf{d}_1 + \overline{\theta}^3 \mathbf{d}_4) + \mathbf{g}^1 \otimes (\mathbf{d}_1 + \overline{\theta}^3 \mathbf{d}_5) + \mathbf{g}^3 \otimes (\mathbf{d}_3 + \theta^1 \mathbf{d}_4 + \theta^2 \mathbf{d}_5)] \mathrm{d}v^*. \tag{B.6}$$

Thus, with the help of Eq. (A.2), it follows that $\mathbf{T}$ is the average of the Cauchy stress

$$d^{1/2}\mathbf{T} = \int_P \mathbf{T}^* \mathrm{d}v^*. \tag{B.7}$$

In order to understand the physical meaning of the director couples (B.3), it is first noted that the quantities $\mathbf{m}_\alpha^0$ represent the total resultant forces applied to the ends ∂P_α. Also, since $\theta^3 = \xi_\alpha$ on ∂P_α, it follows that the quantities $\mathbf{m}_\alpha^i$ are related to each other by the formulas

$$\begin{aligned} &\mathbf{m}_1^3 = -\frac{L}{2}\mathbf{m}_1^0, \qquad \mathbf{m}_2^3 = \frac{L}{2}\mathbf{m}_2^0, \\ &\mathbf{m}_1^4 = -\frac{L}{2}\mathbf{m}_1^1, \qquad \mathbf{m}_2^4 = \frac{L}{2}\mathbf{m}_2^1, \\ &\mathbf{m}_1^5 = -\frac{L}{2}\mathbf{m}_1^2, \qquad \mathbf{m}_2^5 = \frac{L}{2}\mathbf{m}_2^2, \\ &L = \xi_2 - \xi_1. \end{aligned} \tag{B.8}$$

Furthermore, it is noted that the total moments applied to the ends ∂P_α about the origin can be expressed in the forms

$$\begin{aligned} \int_{\partial P_1} \mathbf{x}^* \times \mathbf{t}^* \mathrm{d}a^* &= \left(\mathbf{d}_0 - \frac{L}{2}\mathbf{d}_3\right) \times \mathbf{m}_1^0 + \mathbf{m}_1, \\ \int_{\partial P_2} \mathbf{x}^* \times \mathbf{t}^* \mathrm{d}a^* &= \left(\mathbf{d}_0 + \frac{L}{2}\mathbf{d}_3\right) \times \mathbf{m}_2^0 + \mathbf{m}_2, \end{aligned} \tag{B.9}$$

where $\mathbf{m}_\alpha$ are the moments applied to the ends ∂P_α about the centroids $\theta^\alpha = 0$ of those ends and are defined by expression (75).

Appendix C. Three-dimensionally homogeneous deformations

The objective of this appendix is to develop an expression for the three-dimensional deformation gradient $\mathbf{F}^*$ and to determine the necessary and sufficient conditions for the deformation to be three-dimensionally homogeneous. Also, specific expressions for the resultant forces and couples will be obtained for a three-dimensionally homogeneous nonlinear elastic material. To this end, it is convenient to introduce the tensors

$$\begin{aligned} &\mathbf{\Lambda}_1 = \mathbf{D}_4 \otimes \mathbf{D}^3, \qquad \mathbf{\Lambda}_2 = \mathbf{D}_5 \otimes \mathbf{D}^3, \qquad \mathbf{\Lambda}_3 = \mathbf{D}_4 \otimes \mathbf{D}^1 + \mathbf{D}_5 \otimes \mathbf{D}^2, \\ &\boldsymbol{\lambda}_1 = \mathbf{F}^{-1}\mathbf{d}_4 \otimes \mathbf{D}^3, \qquad \boldsymbol{\lambda}_2 = \mathbf{F}^{-1}\mathbf{d}_5 \otimes \mathbf{D}^3, \qquad \boldsymbol{\lambda}_3 = \mathbf{F}^{-1}(\mathbf{d}_4 \otimes \mathbf{D}^1 + \mathbf{d}_5 \otimes \mathbf{D}^2), \end{aligned} \tag{C.1}$$

so that the covariant and contravariant vectors defined in Appendix A can be written in the forms

$$\mathbf{G}_i = (\mathbf{I} + \theta^\alpha \mathbf{\Lambda}_\alpha + \bar{\theta}^3 \mathbf{\Lambda}_3)\mathbf{D}_i, \qquad \mathbf{G}^i = (\mathbf{I} + \theta^\alpha \mathbf{\Lambda}_\alpha + \bar{\theta}^3 \mathbf{\Lambda}_3)^{-\mathrm{T}}\mathbf{D}^i,$$
$$\mathbf{g}_i = \mathbf{F}(\mathbf{I} + \theta^\alpha \boldsymbol{\lambda}_\alpha + \bar{\theta}^3 \boldsymbol{\lambda}_3)\mathbf{D}_i \qquad \text{for } i = 1,2,3. \tag{C.2}$$

Then, the three-dimensional deformation gradient $\mathbf{F}^*$ can be expressed as

$$\mathbf{F}^* = \sum_{i=1}^{3} \mathbf{g}_i \otimes \mathbf{G}^i = \mathbf{F}(\mathbf{I} + \theta^\alpha \boldsymbol{\lambda}_\alpha + \bar{\theta}^3 \boldsymbol{\lambda}_3)(\mathbf{I} + \theta^\alpha \mathbf{\Lambda}_\alpha + \bar{\theta}^3 \mathbf{\Lambda}_3)^{-1}. \tag{C.3}$$

For three-dimensionally homogeneous deformations, the deformation gradient is independent of the coordinates θ^i so that

$$\mathbf{F}^* = \mathbf{F}(t). \tag{C.4}$$

Thus, with the help of Eq. (C.3), it follows that

$$\boldsymbol{\lambda}_\alpha = \mathbf{\Lambda}_\alpha, \qquad \boldsymbol{\lambda}_3 = \mathbf{\Lambda}_3, \qquad \mathbf{d}_4 = \mathbf{F}\mathbf{D}_4, \qquad \mathbf{d}_5 = \mathbf{F}\mathbf{D}_5. \tag{C.5}$$

This also means that for three-dimensionally homogeneous deformations, the strains $\boldsymbol{\beta}_\alpha$ defined by Eq. (18) necessarily vanish

$$\boldsymbol{\beta}_\alpha = 0. \tag{C.6}$$

Thus, condition (C.6) is a necessary condition for the deformation to be three-dimensionally homogeneous. Moreover, by using Eq. (18) it is easy to see that condition (C.6) is also a sufficient condition because they immediately yield the results (C.5), which can be used in Eq. (C.3) to show that Eq. (C.4) holds.

For a three-dimensional uniform homogeneous nonlinear elastic material, the reference mass density ρ_0^* is constant and it can be shown that the Cauchy stress $\mathbf{T}^*$ and the three-dimensional strain energy function Σ^* can be written in the forms

$$\Sigma^* = \Sigma^*(\mathbf{C}^*), \qquad \mathbf{C}^* = \mathbf{F}^{*\mathrm{T}}\mathbf{F}^*, \qquad J^*\mathbf{T}^* = 2\rho_0^*\mathbf{F}^* \frac{\partial \Sigma^*(\mathbf{C}^*)}{\partial \mathbf{C}^*}\mathbf{F}^{*\mathrm{T}}. \tag{C.7a–c}$$

It then follows that for three-dimensionally homogeneous deformations

$$\mathbf{F}^* = \mathbf{F}, \qquad J^* = J, \qquad \mathbf{C}^* = \mathbf{C}, \qquad J\mathbf{T}^* = 2\rho_0^*\mathbf{F} \frac{\partial \Sigma^*(\mathbf{C})}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}}. \tag{C.8}$$

Also, since ρ_0^* is constant, it follows from Eqs. (A.3) and (B.1) that

$$m = \rho_0^* V D^{1/2}, \qquad D^{1/2} V = \int_{P_0} \mathrm{d}V^*, \tag{C.9}$$

where V is related to the reference volume of the Cosserat point, $\mathrm{d}V^*$ is the element of volume in the reference configuration, and P_0 is the region occupied by the point in the reference configuration. Next, with the help of the fact that

$$\mathbf{g}^i = \mathbf{F}^{*-\mathrm{T}}\mathbf{G}^i \quad \text{for } i = 1,2,3, \tag{C.10}$$

and using expressions (16), (B.5) and (B.7), it can be shown that

$$d^{1/2}\hat{\mathbf{T}} = JVD^{1/2}\mathbf{T}^* = 2m\mathbf{F}\frac{\partial \Sigma^*(\mathbf{C})}{\partial \mathbf{C}}\mathbf{F}^{\mathrm{T}},$$
$$\hat{\mathbf{t}}^4 = 2m\mathbf{F}\frac{\partial \Sigma^*(\mathbf{C})}{\partial \mathbf{C}}\mathbf{V}^1, \qquad \hat{\mathbf{t}}^5 = 2m\mathbf{F}\frac{\partial \Sigma^*(\mathbf{C})}{\partial \mathbf{C}}\mathbf{V}^2, \tag{C.11}$$

where $(\hat{\mathbf{T}}, \hat{\mathbf{t}}^4, \hat{\mathbf{t}}^5)$ are the elastic parts of $(\mathbf{T}, \mathbf{t}^4, \mathbf{t}^5)$. Also, the constant vectors $\mathbf{V}^\alpha$ are defined by

$$D^{1/2}V\mathbf{V}^1 = \int_{P_0} [\mathbf{G}^1\bar{\theta}^3 + \mathbf{G}^3\theta^1]\mathrm{d}V^*,$$

$$D^{1/2}V\mathbf{V}^2 = \int_{P_0} [\mathbf{G}^2\bar{\theta}^3 + \mathbf{G}^3\theta^2]\mathrm{d}V^*. \tag{C.12}$$

Appendix D. Galerkin procedure for determining material constants

The objective of this appendix is to determine material constants that are consistent with the Galerkin procedure applied to the solution of the linearized theory. To this end, consider a beam, whose reference configuration is characterized by Eqs. (77) and (78), and introduce the displacement vectors $\boldsymbol{\delta}_i(t)$ such that

$$\mathbf{d}_0 = \frac{L}{2}\mathbf{e}_3 + \boldsymbol{\delta}_0, \qquad \mathbf{d}_\alpha = \mathbf{e}_\alpha + \boldsymbol{\delta}_\alpha, \qquad \mathbf{d}_3 = \mathbf{e}_3 + \boldsymbol{\delta}_3, \qquad \mathbf{d}_4 = \boldsymbol{\delta}_4, \qquad \mathbf{d}_5 = \boldsymbol{\delta}_5. \tag{D.1}$$

Now, using representations (A.1) and (A.2) for the three-dimensional position vector, and definition (C.3) for the deformation gradient $\mathbf{F}^*$, it follows that

$$\begin{aligned}\mathbf{x}^*(\theta^i,t) &= \mathbf{X}^*(\theta^i) + \boldsymbol{\delta}_0 + \theta^1[\boldsymbol{\delta}_1 + \bar{\theta}^3\boldsymbol{\delta}_4] + \theta^2[\boldsymbol{\delta}_2 + \bar{\theta}^3\boldsymbol{\delta}_5] + \bar{\theta}^3\boldsymbol{\delta}_3, \mathbf{F}^* \\ &= \mathbf{I} + (\boldsymbol{\delta}_1\otimes\mathbf{e}_1 + \boldsymbol{\delta}_2\otimes\mathbf{e}_2 + \boldsymbol{\delta}_3\otimes\mathbf{e}_3) + \theta^1(\boldsymbol{\delta}_4\otimes\mathbf{e}_3) + \theta^2(\boldsymbol{\delta}_5\otimes\mathbf{e}_3) + \bar{\theta}^3(\boldsymbol{\delta}_4\otimes\mathbf{e}_1 + \boldsymbol{\delta}_5\otimes\mathbf{e}_2).\end{aligned} \tag{D.2}$$

Moreover, neglecting quadratic terms in the displacements $\boldsymbol{\delta}_i$ and using definition (C.7b), it can be shown that the linearized form of the three-dimensional Lagrangian strain $\mathbf{E}^*$ becomes

$$\begin{aligned}\mathbf{E}^* &= \tfrac{1}{2}(\mathbf{C}^* - \mathbf{I}) = \mathbf{E} + \theta^1\mathbf{E}_1 + \theta^2\mathbf{E}_2 + \bar{\theta}^3\mathbf{E}_3, \\ \mathbf{E} &= \tfrac{1}{2}[(\boldsymbol{\delta}_1\otimes\mathbf{e}_1 + \boldsymbol{\delta}_2\otimes\mathbf{e}_2 + \boldsymbol{\delta}_3\otimes\mathbf{e}_3) + (\mathbf{e}_1\otimes\boldsymbol{\delta}_1 + \mathbf{e}_2\otimes\boldsymbol{\delta}_2 + \mathbf{e}_3\otimes\boldsymbol{\delta}_3)], \\ \mathbf{E}_1 &= \tfrac{1}{2}[(\boldsymbol{\delta}_4\otimes\mathbf{e}_3) + (\mathbf{e}_3\otimes\boldsymbol{\delta}_4)], \qquad \mathbf{E}_2 = \tfrac{1}{2}[(\boldsymbol{\delta}_5\otimes\mathbf{e}_3) + (\mathbf{e}_3\otimes\boldsymbol{\delta}_5)], \\ \mathbf{E}_3 &= \tfrac{1}{2}[(\boldsymbol{\delta}_4\otimes\mathbf{e}_1 + \boldsymbol{\delta}_5\otimes\mathbf{e}_2) + (\mathbf{e}_1\otimes\boldsymbol{\delta}_4 + \mathbf{e}_2\otimes\boldsymbol{\delta}_5)].\end{aligned} \tag{D.3}$$

Next, it is recalled that the strain energy function Σ^* for an isotropic material can be expressed in the form

$$\rho_0^*\hat{\Sigma}^*(\mathbf{E}^*) = \tfrac{1}{2}(K^* - \tfrac{2}{3}\mu^*)(\mathbf{E}^*\cdot\mathbf{I})^2 + \mu^*(\mathbf{E}^*\cdot\mathbf{E}^*). \tag{D.4}$$

Thus, the Galerkin procedure is consistent with the approximation that the strain energy Σ of the Cosserat point is given by

$$m\Sigma = \int_{P_0} \rho_0^*\hat{\Sigma}^*(\mathbf{E}^*)\mathrm{d}V^*, \tag{D.5}$$

where m is determined by Eq. (C.9). Now, in view of specification (77) and (A.1), it can be shown that

$$\begin{gathered}\int_{P_0}\mathrm{d}V^* = V, \qquad \int_{P_0}\theta^1\mathrm{d}V^* = 0, \qquad \int_{P_0}\theta^2\mathrm{d}V^* = 0, \qquad \int_{P_0}\bar{\theta}^3\mathrm{d}V^* = 0, \qquad \int_{P_0}\theta^1\theta^1\mathrm{d}V^* = \frac{VH^2}{12}, \\ \int_{P_0}\theta^2\theta^2\mathrm{d}V^* = \frac{VW^2}{12}, \qquad \int_{P_0}\bar{\theta}^3\bar{\theta}^3\mathrm{d}V^* = \frac{VL^2}{12}, \qquad \int_{P_0}\theta^1\theta^2\mathrm{d}V^* = \int_{P_0}\theta^1\bar{\theta}^3\mathrm{d}V^* = \int_{P_0}\theta^2\bar{\theta}^3\mathrm{d}V^* = 0.\end{gathered} \tag{D.6}$$

Consequently, by substituting approximation (D.3) into Eq. (D.5) and using Eqs. (50) and (D.6), it follows that

$$m\Sigma = m\hat{\Sigma}^*(\mathbf{E}) + m\Psi, \qquad m\Psi = m\left[\frac{H^2}{12}\hat{\Sigma}^*(\mathbf{E}_1) + \frac{W^2}{12}\hat{\Sigma}^*(\mathbf{E}_2) + \frac{L^2}{12}\hat{\Sigma}^*(\mathbf{E}_3)\right]. \tag{D.7a, b}$$

Furthermore, using definitions (18), (61), specification (78), and linearized expressions (D.1) and (D.3), it can be shown that

$$\boldsymbol{\delta}_4 = \frac{\kappa_1^i}{L}\mathbf{e}_i, \qquad \boldsymbol{\delta}_5 = \frac{\kappa_2^i}{L}\mathbf{e}_i, \qquad \mathbf{E}_\alpha = \frac{\kappa_\alpha^i}{2L}(\mathbf{e}_i \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_i),$$

$$\mathbf{E}_3 = \frac{1}{2L}[\kappa_1^i(\mathbf{e}_i \otimes \mathbf{e}_1 + \mathbf{e}_1 \otimes \mathbf{e}_i) + \kappa_2^i(\mathbf{e}_i \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_i)]. \tag{D.8}$$

Moreover, using Eq. (65), it follows that

$$\begin{aligned}
&(\mathbf{E}_1 \cdot \mathbf{I})^2 = \frac{1}{L^2}(\kappa_1^3)^2, \qquad \mathbf{E}_1 \cdot \mathbf{E}_1 = \frac{1}{2L^2}[(\kappa_1^1)^2 + 2(\kappa_1^3)^2 + (\omega_1 + \omega_2)^2],\\
&(\mathbf{E}_2 \cdot \mathbf{I})^2 = \frac{1}{L^2}(\kappa_2^3)^2, \qquad \mathbf{E}_2 \cdot \mathbf{E}_2 = \frac{1}{2L^2}[(\kappa_2^2)^2 + 2(\kappa_2^3)^2 + (\omega_2 - \omega_1)^2],\\
&(\mathbf{E}_3 \cdot \mathbf{I})^2 = \frac{1}{L^2}[(\kappa_1^1)^2 + 2\kappa_1^1\kappa_2^2 + (\kappa_2^2)^2],\\
&\mathbf{E}_3 \cdot \mathbf{E}_2 = \frac{1}{2L^2}[2(\kappa_1^1)^2 + 2(\kappa_2^2)^2 + (\kappa_1^3)^2 + (\kappa_2^3)^2 + 4\omega_2^2].
\end{aligned} \tag{D.9}$$

Thus, with the help of Eq. (D.9), expression (D.7b) can be written in the form (first part) (65), where the values of the constants k_i associated with the Galerkin procedure are given by

$$\begin{aligned}
&k_1 = \frac{1}{12}\left[\mu^* + \left(K^* + \frac{4}{3}\mu^*\right)\frac{H^2}{L^2}\right], \qquad k_2 = \frac{1}{12}\left[\mu^* + \left(K^* + \frac{4}{3}\mu^*\right)\frac{W^2}{L^2}\right],\\
&k_3 = \frac{1}{12}\mu^*\left[\frac{H^2}{L^2} + \frac{W^2}{L^2}\right], \qquad k_4 = \frac{1}{12}\left[\left(K^* + \frac{4}{3}\mu^*\right) + \mu^*\frac{H^2}{L^2}\right],\\
&k_5 = \frac{1}{12}\left[\left(K^* + \frac{4}{3}\mu^*\right) + \mu^*\frac{W^2}{L^2}\right], \qquad k_6 = \frac{1}{12}\mu^*\left[4 + \frac{H^2}{L^2} + \frac{W^2}{L^2}\right],\\
&k_7 = \frac{1}{6}\left[K^* - \frac{2}{3}\mu^*\right], \qquad k_8 = \frac{1}{6}\mu^*\left[\frac{H^2}{L^2} - \frac{W^2}{L^2}\right].
\end{aligned} \tag{D.10}$$

The values of k_1 and k_2 in Eq. (D.10) are different form the values given in Eq. (100) because kinematic assumption (D.2) is incompatible with uniaxial stress for pure bending. For this reason, the coefficients for bending are usually obtained by modifying the three-dimensional strain energy function and considering only the strain energy of uniaxial stress such that

$$\rho_0^*\bar{\Sigma}^*(\mathbf{E}^*) = \tfrac{1}{2}E^*[\mathbf{E}^* \cdot (\mathbf{e}_3 \otimes \mathbf{e}_3)]^2. \tag{D.11}$$

Then, using kinematic approximation (D.3), it can be shown that

$$\begin{aligned}
&m\Sigma = \int_{P_0} \rho_0^*\bar{\Sigma}^*(\mathbf{E}^*)\,\mathrm{d}V^* = m\bar{\Sigma}^*(\mathbf{E}) + m\Psi,\\
&m\Psi = m\left[\frac{H^2}{12}\bar{\Sigma}^*(\mathbf{E}_1) + \frac{W^2}{12}\bar{\Sigma}^*(\mathbf{E}_2)\right] = \frac{1}{2}E^*\left[\frac{H^2}{12L^2}(\kappa_1^3)^2 + \frac{W^2}{12L^2}(\kappa_2^3)^2\right],
\end{aligned} \tag{D.12}$$

which yields expression (100) for the bending coefficients k_1 and k_2. However, constitutive equation (D.11) does not explicitly model shear deformation or stretching and shearing of the cross-section.

Furthermore, to make the standard Galerkin procedure more apparent, it is desirable to write the three-dimensional position vector in terms of nodal values and shape functions. Specifically, the nodal quantities are the vector ${}_I\mathbf{d}_0^*(t)$, which locates the centroid of the Ith cross-section, and the vectors ${}_I\mathbf{d}_1^*(t)$ and ${}_I\mathbf{d}_2^*(t)$, which can be identified with material fibers in the cross-section. Moreover, the value of θ^3 associated with the Ith cross-section is given by

$$\theta^3 = \xi_I \quad \text{for the } I\text{th cross-section.} \tag{D.13}$$

Next, in order to represent the shape functions associated with these nodal quantities, it is convenient to introduce the functions $\phi_I(\theta^3)$ defined by

$$\begin{aligned}
&\phi_I(\theta^3) = \frac{\xi_2 - \theta^3}{\xi_2 - \xi_1} \quad \text{for } \xi_1 \leqslant \theta^3 \leqslant \xi_2, \qquad \phi_I(\theta^3) = 0 \quad \text{for } \theta^3 \geqslant \xi_2,\\
&\phi_I(\theta^3) = \frac{\theta^3 - \xi_{I-1}}{\xi_I - \xi_{I-1}} \quad \text{for } \xi_{I-1} \leqslant \theta^3 \leqslant \xi_\mathrm{I}, \qquad \phi_I(\theta^3) = \frac{\xi_{I+1} - \theta^3}{\xi_{I+1} - \xi_I} \quad \text{for } \xi_I \leqslant \theta^3 \leqslant \xi_{I+1},\\
&\phi_I(\theta^3) = 0 \qquad \text{for } \theta^3 \leqslant \xi_{I-1} \quad \text{or } \theta^3 \geqslant \xi_{I+1}, \quad \text{for } I = 2, 3, \ldots, N\\
&\phi_{N+1}(\theta^3) = \frac{\theta^3 - \xi_N}{\xi_{N+1} - \xi_N} \quad \text{for } \xi_N \leqslant \theta^3 \leqslant \xi_{N+1}, \qquad \phi_{N+1}(\theta^3) = 0 \quad \text{for } \theta^3 \leqslant \xi_N,
\end{aligned} \tag{D.14}$$

which have the usual properties that

$$\phi_I(\xi_J) = \delta_{IJ} \quad \text{for } I, J = 1, 2, \ldots, N+1. \tag{D.15}$$

Finally, the position vector in the rod-like region is expressed in the form

$$\mathbf{x}^*(\theta^i, t) = \sum_{I=1}^{N+1} \phi_I(\theta^3)[{}_I\mathbf{d}_0^*(t) + \theta^1{}_I\mathbf{d}_1^*(t) + \theta^2{}_I\mathbf{d}_2^*(t)], \tag{D.16}$$

where the shape function are given by

$$\{\phi_I, \phi_I\theta^1, \phi_I\theta^2\}. \tag{D.17}$$

Although higher order dependence on θ^α can be considered in the general Galerkin method, these shape functions have been specified to be consistent with kinematic assumption (A.2) associated with Cosserat theory.

Appendix E. Basic equations for the elastica

The elastica is a model for the nonlinear deformation of an elastic rod that neglects extension along the rod, tangential shear deformation, normal cross-sectional extension and normal cross-sectional shear deformation. The equations for the elastica have been reviewed by Love (1944), but here it is more convenient to use the notation presented in Rubin (1997), where a generalized intrinsic formulation for nonlinear elastic rods has been presented.

For two-dimensional static deformation of the rod, the reference curve is characterized by the Lagrangian coordinate θ^3 and the position vector $\mathbf{x}(\theta^3)$, such that

$$\mathbf{x} = x_1(\theta^3)\mathbf{e}_1 + x_3(\theta^3)\mathbf{e}_3, \quad 0 \leqslant \theta^3 \leqslant L_0. \tag{E.1}$$

Also, the unit tangent vector $\mathbf{e}_\mathrm{t}$ and unit normal vector $\mathbf{e}_\mathrm{n}$ are defined by

$$\mathbf{e}_t = \frac{\partial \mathbf{x}}{\partial \theta^3} = \sin\theta \mathbf{e}_1 + \cos\theta \mathbf{e}_3, \qquad \mathbf{e}_n = \mathbf{e}_2 \times \mathbf{e}_t = \cos\theta \mathbf{e}_1 - \sin\theta \mathbf{e}_3, \tag{E.2a, b}$$

where $\theta(\theta^3)$ is the angle that the tangent vector $\mathbf{e}_t$ makes with the direction $\mathbf{e}_3$. Next, the force $\mathbf{n}$ and moment $\mathbf{m}$ acting on the cross-section, whose unit outward normal is in the $\mathbf{e}_t$ direction are expressed in the forms

$$\begin{aligned} &\mathbf{n} = n\mathbf{e}_t + v\mathbf{e}_n, \qquad \mathbf{m} = m\mathbf{e}_2, \qquad v = -\frac{\mathrm{d}m}{\mathrm{d}\theta^3}, \\ &\mathrm{m} = \frac{\mathrm{d}\sigma}{\mathrm{d}\kappa} = E^* I^* \kappa, \qquad I^* = \frac{H^3 W}{12}, \qquad \sigma = \frac{1}{2} E^* I^* \kappa^2, \qquad \kappa = \frac{\mathrm{d}\theta}{\mathrm{d}\theta^3}, \end{aligned} \tag{E.3}$$

where n is an arbitrary function of θ^3, which is a constraint response associated with the inextensibility condition, I^* is determined by the geometry of the cross-section, κ is the curvature of the center line of the beam, and σ is the strain energy function associated with bending. Here, the unit normal vector $\mathbf{e}_n$ is defined by Eq. (E.2b) instead of by the usual expression given in Rubin (1997). Therefore, $\mathbf{e}_n$ does not necessarily point towards the inside of the curve so that κ can have positive or negative values.

In the absence of body forces and surface traction on the lateral surface of the beam, the equilibrium equation reduces to

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\theta^3} = 0. \tag{E.4}$$

Thus, for the problem described in Section 10, it can be shown that the solution of Eq. (E.4) can be expressed in the form

$$\begin{aligned} &\mathbf{n} = P(\sin\alpha \mathbf{e}_1 - \cos\alpha \mathbf{e}_3) = P[-\cos(\theta+\alpha)\mathbf{e}_t + \sin(\theta+\alpha)\mathbf{e}_n], \\ &n = -P\cos(\theta+\alpha), \qquad v = P\sin(\theta+\alpha), \end{aligned} \tag{E.5}$$

where P and α are constants. Now, Eqs. (E.2a) and (E.3) (third part) can be rewritten as

$$\frac{\mathrm{d}x_1}{\mathrm{d}\theta^3} = \sin\theta, \qquad \frac{\mathrm{d}x_3}{\mathrm{d}\theta^3} = \cos\theta, \qquad \frac{\mathrm{d}^2\theta}{\mathrm{d}(\theta^3)^2} + \left[\frac{P}{E^* I^*}\right]\sin\theta = 0. \tag{E.6}$$

Moreover, for the problem of Section 10, these equations are integrated subject to the boundary conditions

$$x_1(0) = 0, \qquad x_3(0) = 0, \qquad \theta(0) = 0, \qquad \frac{\mathrm{d}\theta}{\mathrm{d}\theta^3}(L_0) = 0. \tag{E.7}$$

Specifically, Eq. (E.6) is integrated using the program MATLAB by guessing the value $\kappa(0)$ of the curvature at the end $\theta^3 = 0$

$$\frac{\mathrm{d}\theta}{\mathrm{d}\theta^3}(0) = \kappa(0), \tag{E.8}$$

and iterating on the value of $\kappa(0)$ until fourth part of condition (E.7) (requiring the moment to vanish) is satisfied.

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